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Classical and quantum aspects of the extended antifield formalism

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Abstract

Starting from a solution to the classical Batalin-Vilkovisky master equation, an extended solution to an extended master equation is constructed by coupling all the observables, the anomaly candidates and the generators of global symmetries. The construction of the formalism and its applications in the context of the renormalization of generic and potentially anomalous gauge theories are reviewed.

The main aim of the standard antifield formalism is the construction, for generic gauge theories, of the proper solution of the master equation. The master equation is formulated in terms of the antisymplectic structure for the fields and antifields, and its solution is the generator of the BRST differential. The coupling constants play a passive role in the usual discussions. The most important feature of the extended antifield formalism is to promote the coupling constants to active participants of the construction, by allowing the BRST differential to act on them, and by introducing, in an intermediate stage “anticoupling constants” which are their partners in an extended antibracket. The understanding that the physically relevant coupling constants are related to independent local BRST cohomology classes in ghost number 0 naturally leads to consider the constant ghosts coupled to the generators of generalized global symmetries (local BRST cohomology classes in negative ghost numbers) as generalized coupling constants and to include the couplings to anomaly and anomaly for anomaly candidates (local BRST cohomology classes in positive ghost numbers).

The heart of the extended antifield formalism is the construction of an extended master equation for an extended action to which all these cohomology classes have been coupled and a characterization of the cohomology of the associated BRST differential, which takes into account in a systematic way higher order cohomological restrictions through the Lie-Massey brackets. This construction is shown to be a particular case of a general structure that is available as soon as one has a differential graded Lie algebra, i.e., a graded vector space with an even or odd Lie bracket and a differential that is a graded derivation of the bracket.

Applications of these ideas in the context of renormalized quantum field theory are then discussed. More precisely:

- The existence and relevance of higher order cohomological restrictions on anomalies and counterterms is demonstrated. As an illustration, the case of Yang-Mills theories with abelian factors is discussed.
- It is shown that the use of the extended formalism guarantees stability independently of power counting restrictions (“renormalizability in the modern sense”) for generic gauge theories.
- The anomalous Zinn-Justin equation for the renormalized effective action can be written to all orders as a functional differential equation. This allows to prove the existence of a quantum BRST differential and to extend the whole framework of algebraic renormalization to anomalous gauge theories. In particular, the existence of well defined quantum BRST cohomologies is proved.
- The dependence of the quantum theory on the parameters of the gauge fixing is considered and the general structure of the Callan-Symanzik and the renormalization group equations is discussed.
- A refined anomaly consistency condition for local BRST cohomology classes is derived. As an application, a new approach to the Adler-Bardeen theorem on the non renormalization of the non abelian gauge anomaly is proposed.

This review is based on the papers [1, 2, 3, 4, 5, 6], with the following improvements:

(i) The analysis of [2, 3] is done without the assumption that there are no anomalies.

(ii) The analysis of [6] is rewritten in the context of the extended antifield formalism, so that the assumptions of stability and absence of anomalies are now not necessary. As a consequence, the dependence of the anomaly coefficients on the gauge parameters and the renormalization scale is obtained.

(iii) A new section on the quantum BRST cohomology groups has been added.

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1 Introduction

Yang-Mills theories

The best known example of renormalization of a theory invariant under a non linear symmetry is probably non abelian Yang-Mills theory: on the level of the gauge fixed Faddeev-Popov action [7], gauge invariance is expressed through invariance under the non linear global BRST symmetry [8, 9, 10, 11, 12].

Some of the crucial points in the analysis are: (i) the importance of the BRST cohomology as a constraint on the anomalies and the counterterms of the theory, (ii) the anticanonical structure of the theory in terms of the fields and the sources, to which the BRST variations are coupled, together with the compact reformulation of all the Ward identities in terms of the Zinn-Justin equation [13, 14], and (iii) the insight that BRST exact counterterms can be absorbed by anticanonical fields and sources redefinitions [15].

The question whether the remaining counterterms can be absorbed by a redefinition of the coupling constants of the theory could be settled to the affirmative, in the power counting renormalizable case based on a semi-simple gauge group, through an exhaustive enumeration of all possible renormalizable interactions [10]. In the case where one includes higher dimensional gauge invariant operators, such a property depends crucially on a conjecture by Kluberg-Stern and Zuber [16] on the BRST cohomology in ghost number 0, which states that it should be describable by off-shell gauge invariant operators not involving the ghosts or the sources. This conjecture can be proved [17, 18] in the semi-simple case for which it has been originally formulated, but it is not valid in the presence of abelian factors, not even for power counting renormalizable theories [19, 20, 17, 18]. In this last case, the counterterms violating the generalized Kluberg-Stern and Zuber conjecture have been shown to be absent by more involved arguments from renormalization theory [19, 20], so that renormalizability still holds, even if the conjecture does not.

Extended antifield formalism

The classical Batalin-Vilkovisky formalism [21, 22, 23, 24, 25] (for reviews, see e.g. [26, 27]) extends the above techniques to the case of general gauge theories with open gauge algebras and structure functions, the invariance of the action being expressed through the central master equation.

These cohomological techniques can also be used in order to control the renormalization of non linear global symmetries with a closed algebra by coupling their generators with constant ghosts [28, 29]. The generalization of the formalism to the case of systems including both gauge and generalized global symmetries with generic algebra has been achieved only recently [30, 31]. It is based on the observation that the generators of generalized global symmetries correspond to local BRST cohomological classes in negative ghost numbers [32] (for a review, see [33]).

Stability in the BV formalism

A detailed analysis of the compatibility of the renormalization procedure with invariance expressed through the master equation has been performed in [34, 35, 36, 37], where it has been shown that the renormalized action is a deformation of the starting point solution to the master equation. Independently of this result, the fundamental problem of locality of the construction is raised and a locality hypothesis is stated [34]. This hypothesis can be reinterpreted in a more general framework as the assumption that the cohomology of the Koszul-Tate differential [38, 39] vanishes in the space of local functionals. While the assumption holds under certain conditions, which are in particular fulfilled for the construction of the solution of the master equation, thus guaranteeing its locality [39], it does not hold in general: the obstructions are related to the non trivial global currents of the theory [32], and give rise to BRST cohomology classes with a non trivial antifield dependence.

A consequence is the possibility of existence of observables that cannot be made off-shell gauge invariant, even in the case of closed gauge theories, so that the associated deformed solutions of the master equation cannot be related by a field, antifield and coupling constant renormalization to the starting point solution extended by coupling all possible off-shell observables.

In [40, 41], renormalization in the context of the Batalin-Vilkovisky formalism is reconsidered precisely under the assumption that there are no such deformations, i.e., in the closed case under the analog of the Kluberg-Stern and Zuber conjecture, and in the open case under the conjecture that all the BRST cohomology is already contained in the solution to the master equation coupled with independent coupling constants¹, with the conclusion, that the infinities can then be absorbed by renormalizations.

Finally, in [42] the problem of renormalization under non linear symmetries is readressed in the context of effective field theories: it is for instance shown that semi-simple Yang-Mills theory and gravity, to which are coupled all possible (power counting non renormalizable) off-shell observables, are such that all the local counterterms needed to cancel the infinities, can be absorbed through coupling constants, field and antifield renormalizations, while preserving the symmetry (in the form of the Batalin-Vilkovisky master equation). Theories possessing this last property, even if an infinite number of coupling constants is needed, are defined to be renormalizable in the modern sense. The difficulty, that is also discussed, is that the non trivial infinities are a priori only constrained to belong to the BRST cohomology in ghost number 0, which, because of the non validity of the generalized Kluberg-Stern and Zuber conjecture (taken as an example of a so called structural constraint), does not guarantee that they can be absorbed by redefinitions of coupling constants of an action extended by all possible off-shell observables. What good structural constraints might be in the general case and if they can be chosen in such a way as to guarantee renormalizability in the modern sense for all theories is left as an open question in [42, 43].

In order to relate the terminology in [42, 43] to the one coming from the algebraic

¹Note that it is not true that the antifield independent part of the cohomology of the differential (S, \cdot) is off-shell gauge invariant, it is in general only weakly gauge invariant.

approach to renormalization under symmetries pioneered in [44, 8, 9, 10, 11] (see for instance [45, 46, 47] for reviews), we note that what is called (extended) master equation in the former corresponds to a generalized Slavnov-Taylor identity in the latter. By generalized, we mean the definition of this identity in theories with arbitrary gauge structure as proposed by Batalin and Vilkovisky, then the extension to include the case of (a closed subset of) global symmetries. What is called (local) BRST cohomology corresponds to the cohomology of the generalized linearized Slavnov-Taylor operator S_L acting in the space of (integrated) polynomials in the fields, the sources and their derivatives. The question of *renormalizability in the modern sense* in [42, 43] is then the question of *stability independently of power counting restrictions* in the language of algebraic renormalization².

Higher order cohomological restrictions

A clue to the answer to these questions can be found in [34, 35, 36, 37]. Indeed, the fact that the divergences are such that they always provide a deformation of the solution of the master equation, implies in general that the non trivial first order deformations satisfy additional cohomological restrictions [48] besides belonging to the BRST cohomology. The problem with these additional restrictions is that they are non linear. The starting point for the extended antifield formalism to be discussed in this review is to show that the non trivial counterterms and anomalies satisfy linear higher order cohomological constraints, by coupling arbitrary BRST cocycles to the solution of the master equation.

As an (academical) example of how these higher order cohomological restrictions work, consider Yang-Mills theories with free³ abelian gauge fields A_μ^a as discussed for instance in [42]. The BRST cohomology in ghost number zero contains the term [17, 18]

$$K = f_{abc} \int d^n x F^{a\nu\mu} A_\mu^b A_\nu^c + 2A^{*a\mu} A_\mu^b C^c + C^{*a} C^b C^c,$$

for completely antisymmetric constants f_{abc} , so that this term is a potential counterterm. At the same time, the term $k^d \int d^n x C_d^*$ belongs to the BRST cohomology in ghost number -2 . If we take the action $S_k = S + k^d \int d^n x C_d^*$, we have $1/2(S_k, S_k) = O(k^2)$. This implies according to the quantum action principle for the regularized theory that $1/2(\Gamma_k, \Gamma_k) = O(k^2)$ and then, at order 1 in \hbar for the divergent part, that

$$(S_k, \Gamma_k^{(1)}{}_{div}) = O(k^2).$$

The k independent part of this equation gives the usual condition that the divergent part of the k independent effective action at first order must be BRST closed,

²Stability is defined for instance in [46] as “the dimension of the cohomology space of the S_L operator in the Faddeev-Popov neutral sector should be equal to the number of physical parameters of the classical action”.

³By free, we mean that the abelian gauge fields have no couplings to matter fields, hence, they have no interactions at all. Their quantization is of course trivial and we know a priori that no counterterms are needed.

$(S, \Gamma^{(1)}_{div}) = 0$, and contains in particular the candidate K above. The k linear part of this equation requires

$$\left(\frac{\partial \Gamma^{(1)}_{k \text{ div}}}{\partial k^d} \right) \Big|_{k=0}, S) + \left(\int d^n x C_d^*, \Gamma^{(1)}_{div} \right) = 0.$$

This condition eliminates the candidate K because

$$\left(\int d^n x C_d^*, K \right) = 2f_{abd} \int d^n x A^{*a\mu} A_\mu^b + C^{*a} C^b$$

is not BRST exact but represents a non trivial BRST cohomology class in ghost number -1 . Hence, there exists a purely cohomological reason why K cannot appear as a counterterm. Note that as soon as the abelian fields are coupled to matter fields, the functionals $\int d^n x C_d^*$ but also K do not belong to the BRST cohomology any more and the problem with this particular type of counterterms does not arise to begin with.

Anomalies

A related problem, which is relevant in [37, 40, 41, 42], is to provide a sensible definition of the Δ operator of the quantum Batalin-Vilkovisky master equation. Indeed, its expression as a second order functional differential operator with respect to fields and antifields, obtained from formal path integral considerations, does not make sense when applied to local functionals. In [49, 50, 51, 52, 53], the antifield formalism has been discussed in the context of explicit regularization and renormalization schemes and the related question of anomalies (assumed to be absent in [37, 40, 41, 42]) has been addressed. In particular, non trivial anomalies are shown to be constrained by the local BRST cohomology in ghost number 1, which is the expression of the Wess-Zumino consistency condition [66] in this context, and well defined expressions for the regularized Δ operator are proposed at one loop level in [49] in the context of Pauli-Villars regularization and at higher orders in [52, 53] for non-local regularization.

In the context of dimensional renormalization, it has been shown in [54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65] that anomalies can be dealt with consistently, if evanescent terms are taken into account properly. In particular [65], the evanescent breaking terms of the master equation are responsible for a non trivial Δ operator, even though " $\delta(0)$ " = 0. Furthermore, it is shown in [59, 65] that it is useful to couple the non trivial anomalies from the beginning with coupling constants in ghost number -1 in order to control the absorption of divergences in the presence of anomalies.

The completely extended antifield formalism

As a combination of the previous three ideas, i.e., (i) that global symmetries are controlled by coupling BRST cohomological classes in negative ghost numbers, (ii) that renormalizability, or more precisely stability, is related to the question if the

BRST cohomological classes in ghost number 0 are all coupled and (iii) that renormalization in the presence of non trivial anomalies requires the coupling of the local BRST cohomology classes in positive ghost numbers, one is naturally led to consider the solution of the classical master equation to which all local BRST cohomology classes have been coupled. To first order in the couplings, this extension will satisfy the standard master equation. To all orders in the couplings, the equation satisfied by a suitable completion of the extension will satisfy an extended master equation which has the same structure than the extended master equation of the formalism where only the local BRST cohomological classes in negative ghost numbers have been coupled. This will be shown by similar techniques than the ones used in the extended antifield formalism of [31].

A remarkable property of this extended antifield formalism is that the cohomology in all ghost numbers of the BRST differential associated to the extended master equation is contained completely in the solution to this master equation. As a consequence, using this formalism, renormalizability in the modern sense will be shown to hold by construction for all gauge theories, which answers the questions raised in [42, 43], even in the presence of anomalies. In other words, we will complete the general analysis of the absorption of divergences in the presence of possibly anomalous local or global symmetries independently of the precise form of the BRST cohomology of the theory.

At the same time, it will be shown that in the extended formalism, there exist well defined differentials, both on the classical and on the quantum level, that play the role of the quantum Batalin-Vilkovisky Δ operator.

From the point of view of the algebraic approach to renormalization, the extended antifield formalism is stable by construction for all theories. It allows to extend the algebraic approach to the case of anomalous gauge theories. Two well defined isomorphic versions of quantum BRST cohomology will be constructed in this context: the first one on the level of local insertions and the second one on the level of derivations in the couplings of the theory.

Organization of the review

The definition of a differential graded Lie algebra $(L, d, [\cdot, \cdot])$ is recalled in section 2 and the existence of Lie-Massey brackets in cohomology [67] is briefly discussed. Associated to a Hodge decomposition $L = K \oplus R \oplus dR$, with $K \simeq H(d, L)$ and $\{e_a\}$ a basis of K , two additional algebras are considered. If sK^* is the suspension of the dual K^* of K with basis the “couplings” $\{\xi^a\}$, the first one, $L(\xi)$, is the tensor product of L with the polynomials in the couplings, $\wedge(sK^*)$. The second one is the commutator algebra of graded right derivations in the couplings, $RDer[\wedge(sK^*)]$. On $L(\xi)$, an “extended master equation” involving a nilpotent graded right derivation Δ on $\wedge(sK^*)$ is constructed in such a way that the cohomology defined by the solution of the extended master equation in $L(\xi)$ is isomorphic to the commutator cohomology of $[\Delta, \cdot]$ in $RDer[\wedge(sK^*)]$. The perturbative techniques used for the construction are similar to those of [31]. It is then shown how to compute these cohomologies by using a spectral sequence associated to the polynomial degree in the couplings. At the end

of section 2, we discuss some examples of differential graded Lie algebras, where the general construction can be applied.

In section 3, we review relevant features of the antifield formalism, such as: the spectrum of fields and ghosts, the antisymplectic structure between fields and antifields, the master equation, local BRST cohomology and the gauge fixing procedure.

Section 4 is devoted to show that anomalies and divergences must satisfy cohomological restrictions. In order to do so, we use the framework of dimensional regularization as discussed in [65]. The standard restrictions are derived first, while higher order restrictions are obtained by coupling arbitrary BRST cocycles to the action. As an application in the physically relevant case of the standard model, it is shown explicitly how antifield dependent counterterms can be eliminated through higher order cohomological restrictions.

The application of the general construction of section 2 to the case of the (classical) antifield formalism is discussed in detail in section 5. Essential couplings are defined as the couplings corresponding to independent local BRST cohomological classes. The stability of the extended antifield formalism constructed in this way is then proved.

In section 6, the absorption of the divergences in the context of the extended antifield formalism is considered, first in the context of “dimensional” regularization. Using suitable BRST breaking counterterms, the renormalized effective action is shown to satisfy a deformed extended master equation, and this to all orders in \hbar independently of any assumptions on the anomalies of the theory. Alternatively, because stability of the extended antifield formalism has been proved in the previous section on the classical level, the machinery of algebraic renormalization on the final renormalized level can be applied instead of using first a regularization. The deformed extended master equation for the effective action is rederived in this context. Then the renormalization of classical BRST cohomological classes is discussed. At the end of section 6, two well defined quantum BRST cohomologies associated to the deformed extended master equation are introduced, one for local insertions and one for right derivations, and their cohomologies are shown to be isomorphic.

The dependence of the effective action on the parameters introduced through the gauge fixing is analyzed in section 7. It is shown that, by suitably redefining the essential couplings by gauge parameter dependent terms of higher order in \hbar , the variation of the effective action can be made trivial in the sense that it is given by a quantum BRST coboundary, while the anomaly operator can be shown to be independent of the gauge parameters.

In section 8, the general form of the renormalization group equation and the relation between the renormalization group β functions and the anomaly coefficients is derived. If the theory is expressed in terms of the running couplings, we show that the variation of the effective action with respect to the renormalization scale is a quantum BRST coboundary and that the anomaly operator is independent of the renormalization scale. Then, the Callan-Symanzik equation is derived and the vector field built out of the associated β functions is shown to define a non trivial quantum BRST cocycle.

In section 9, the anomaly appearing in the renormalization of a local BRST co-

homological class with a descent of length d and a lift of length l is shown to be characterized by a descent which is shorter or equal to d and a lift which is longer or equal to l . In a first part, the characterization of local BRST cohomological groups according to the lengths of their descents and their lifts is explained. Then, the main result on the lengths of the descents and the lifts of the anomalies is proved. An alternative derivation of the main results clarifying the underlying mechanism is presented in section 9.3, where differentials controlling the one loop anomalies arising in the renormalization of BRST cohomological groups are introduced.

Anomalous effective Yang-Mills theories and a new approach to the Adler-Bardeen theorem, independently of the gauge fixing, of power counting restrictions and without relying on the Callan-Symanzik equation, are discussed in section 10 as an application of the results of section 9.

2 Cohomology of differential graded Lie algebra and associated sh Lie algebra

2.1 Differential graded Lie algebras

Definition Let $L = \bigoplus_{g \in \mathbb{Z}} L^g$ be a graded differential Lie algebra over a field k (typically \mathbb{R} or \mathbb{C}) with an even or an odd bracket:

$$\begin{aligned} d : L^g &\longrightarrow L^{g+1}, \quad d^2 = 0, \\ [\cdot, \cdot] : L^{g_1} \otimes L^{g_2} &\longrightarrow L^{g_1+g_2+\epsilon}, \end{aligned} \quad (2.1)$$

where $\epsilon = -1, 0, 1$ such that

$$[x, y] = -(-)^{(|x|+\epsilon)(|y|+\epsilon)}[y, x], \quad (2.2)$$

$$[x, [y, z]] = [[x, y], z] + (-)^{(|x|+\epsilon)(|y|+\epsilon)}[y, [x, z]], \quad (2.3)$$

$$d[x, y] = [dx, y] + (-)^{|x|+\epsilon}[x, dy], \quad (2.4)$$

for homogeneous elements $x, y, z \in L$.

Remarks :

(i) In principle, it is sufficient to consider the case $\epsilon = 0$, because this case can be obtained from the case $\epsilon \neq 0$ by defining a new grading which is equal to the old one plus ϵ . However, because in various applications it is more natural to have a bracket of degree ϵ , we will keep ϵ in the expressions below

(ii) A particular case is that of an inner differential $d = [D, \cdot]$, where $D \in L^{1-\epsilon}$ satisfies the equation $\frac{1}{2}[D, D] = 0$.

Lie-Massey brackets Because the differential is a derivation of the bracket (2.4), the bracket of 2 cocycles is again a cocycle, and changing one of the cocycles by a coboundary modifies the bracket only by a coboundary. It follows that the bracket $[\cdot, \cdot]$ induces a well defined bracket $[\cdot, \cdot]_M$ in cohomology,

$$\begin{aligned} [\cdot, \cdot]_M : H^{g_1}(d) \otimes H^{g_2}(d) &\longrightarrow H^{g_1+g_2+\epsilon}(d) \\ [[x_1], [x_2]]_M &= [[x_1, x_2]]. \end{aligned} \quad (2.5)$$

There exist also higher order maps induced in cohomology, the Lie-Massey brackets [67]. (In this context, the map $[\cdot, \cdot]_M$ is called the two place Lie-Massey bracket.) For instance, consider cocycles x_i, x_j, x_k , such that $[[x_i], [x_j]]_M = 0$, or, equivalently $[x_i, x_j] = dx_{ij}$. Note that by this equation, the x_{ij} 's are defined only up to cocycles. A three place bracket of such cocycles is defined by

$$[x_i, x_j, x_k] = [x_{ij}, x_k] - (-)^{(g_j+\epsilon)(g_k+\epsilon)}[x_{ik}, x_j] - (-)^{g_i+\epsilon}[x_i, x_{jk}]. \quad (2.6)$$

Because of the derivation property (2.4) and the Jacobi identity (2.3), $d[x_i, x_j, x_k] = 0$. The corresponding cohomology class is defined to be the value of the three place Lie Massey bracket, $[[x_i], [x_j], [x_k]]_M = [[x_i, x_j, x_k]]$. The set of elements $\{x_i, x_{ij}\}$ is called the defining system of $[[x_i], [x_j], [x_k]]_M$. Because the x_{ij} are only defined up to

cocycles, $[[x_i], [x_j], [x_k]]_M$ is not uniquely defined, but it is in fact a set of cohomology classes depending on the defining system. A uniquely defined three place Lie Massey bracket $[[x_i], [x_j], [x_k]]_M$ is obtained for elements $[x_i]$ that belong to the kernel of the two place Lie-Massey bracket, $[x_i] \in \ker_2^g \subset H^g(d)$, iff $[[x_i], [y]]_M = 0$, for all $[y] \in H(d)$. Indeed, in this case, it is straightforward to verify that $[[x_i, x_j, x_k]] \in H^{g_i+g_j+g_k+2\epsilon-1}(d)$ does not depend on the choice of the defining system. Hence, there is a well-defined map

$$\begin{aligned} [\cdot, \cdot, \cdot]_M : \ker_2^{g_1} \otimes \ker_2^{g_2} \otimes \ker_2^{g_3} &\longrightarrow H^{g_1+g_2+g_3+2\epsilon-1}(d) \\ [[x_i], [x_j], [x_k]]_M &= [[x_i, x_j, x_k]]. \end{aligned}$$

Higher order brackets can be systematically defined along similar lines [67]. We will not discuss them here, because we will follow a different route and construct related maps in a systematic way through homological perturbation theory below.

Hodge decomposition and associated graded Lie algebras Suppose one has the Hodge decomposition $L = K \oplus R \oplus dR$, with $K \simeq H(d, L)$ and let $\{e_a\}$ denote the elements of a basis of K . This means that

$$\begin{aligned} dx = 0 &\implies x = \lambda^a e_a + dy, \\ \mu^a e_a = dz &\implies \mu^a = 0 = dz, \end{aligned} \tag{2.7}$$

with $\lambda^a, \mu^a \in k$.

Let us now consider the space $\wedge(sK^*)$, i.e., the exterior algebra over the suspension of the dual K^* of K . In other words, we associate to each e_a a variable ξ^a of grading $\deg(\xi^a) = -|e_a| + 1 - \epsilon$ and take linear combinations $\lambda(\xi) = \sum_k \lambda_{a_1 \dots a_k} \xi^{a_1} \dots \xi^{a_k}$ with $\lambda_{a_1 \dots a_k} \in k$. We also consider the space $L(\xi) \equiv \wedge(sK^*) \otimes L$, with grading given by the sum of the gradings,

$$\text{tot}(\xi^{a_1} \dots \xi^{a_k} x) = \sum_{i=1}^k \deg(\xi^{a_i}) + |x|. \tag{2.8}$$

By abuse of notation, the differential $1 \otimes d$ on $L(\xi)$ is still denoted by d ,

$$d(\xi^{a_1} \dots \xi^{a_k} x) = (-)^{\sum_{i=1}^k \deg(\xi^{a_i})} \xi^{a_1} \dots \xi^{a_k} dx, \tag{2.9}$$

and the bracket is extended according to

$$[\xi^{a_1} \dots \xi^{a_k} x, \xi^{b_1} \dots \xi^{b_m} y] = (-)^{(\sum_{i=1}^m \deg(\xi^{b_i}))(|x|+\epsilon)} \xi^{a_1} \dots \xi^{a_k} \xi^{b_1} \dots \xi^{b_m} [x, y]. \tag{2.10}$$

It follows that $L(\xi)$ is a differential graded Lie algebra with respect to the total degree tot .

Let us also introduce the even graded Lie algebra $R\text{Der}[\wedge(sK^*)]$ of graded right derivations on $\wedge(sK^*)$,

$$R\text{Der}[\wedge(sK^*)] = \left\{ \frac{\partial^R}{\partial \xi^a} \lambda^a(\xi), \lambda^a(\xi) \in \wedge(sK^*) \right\}. \tag{2.11}$$

If $\frac{\partial^R}{\partial \xi^a} \lambda^a(\xi) : \wedge(sK^*)^g \longrightarrow \wedge(sK^*)^{g+\lambda}$, the grading of $\frac{\partial^R}{\partial \xi^a} \lambda^a(\xi)$ is defined to be λ ; the graded commutator is defined by

$$[\frac{\partial^R}{\partial \xi^a} \lambda^a(\xi), \frac{\partial^R}{\partial \xi^b} \mu^b(\xi)]_C = \frac{\partial^R}{\partial \xi^a} (\frac{\partial^R \lambda^a}{\partial \xi^b} \mu^b - (-)^{\lambda \mu} \frac{\partial^R \mu^a}{\partial \xi^b} \lambda^b). \quad (2.12)$$

2.2 Main theorem

For every $n \geq 2$, one can construct, on the one hand, constants $f_{a_1 \dots a_n}^b \in k$, with associated right derivation

$$\Delta = \frac{\partial^R}{\partial \xi^b} f^b(\xi) \in RDer[\wedge(sK^*)]^1, \quad (2.13)$$

where $f^b(\xi) = \sum_{n \geq 2} f_{a_1 \dots a_n}^b \xi^{a_1} \dots \xi^{a_n}$, and, on the other hand, vectors $e_{a_1 \dots a_n} \in L$, with associated element

$$e(\xi) = \sum_{n \geq 1} e_{a_1 \dots a_n} \xi^{a_1} \dots \xi^{a_n} \in L(\xi)^{1-\epsilon}, \quad (2.14)$$

such that:

- the derivation Δ is a differential,

$$\Delta^2 = 0; \quad (2.15)$$

[with the associated left derivation $\Delta^L = (-)^{b+1-\epsilon} f^b(\xi) \frac{\partial^L}{\partial \xi^b} \in LDer[\wedge(sK^*)]^1$ also being a differential, $(\Delta^L)^2 = 0$]

- the element $e(\xi)$ satisfies the Maurer-Cartan type equation

$$(d + \Delta^L)e(\xi) + \frac{1}{2}[e(\xi), e(\xi)] = 0, \quad (2.16)$$

so that

$$\bar{d} = d + [e(\xi), \cdot] + \Delta^L : L(\xi)^g \longrightarrow L(\xi)^{g+1}, \quad \bar{d}^2 = 0; \quad (2.17)$$

- for an inner differential $d = [D, \cdot]$, where $D \in L^{1-\epsilon}$ satisfies the “classical master equation”

$$\frac{1}{2}[D, D] = 0, \quad (2.18)$$

$D(\xi) = D + e(\xi) \in L(\xi)^{1-\epsilon}$ satisfies the “quantum master equation”

$$\frac{1}{2}[D(\xi), D(\xi)] + \Delta D(\xi) = 0, \quad (2.19)$$

and $\bar{d} = [D(\xi), \cdot] + \Delta^L$;

- if $d_\Delta = [\Delta, \cdot]_C$, then

$$H^*(\bar{d}, L(\xi)) \simeq H^*(d_\Delta, RDer[\wedge(sK^*)]), \quad (2.20)$$

the isomorphism being given by

$$H^*(d_\Delta, RDer[\wedge(sK^*)]) \ni \left[\frac{\partial^R}{\partial \xi^a} \lambda^a(\xi) \right] \longleftrightarrow \left[\frac{\partial^R e(\xi)}{\partial \xi^a} \lambda^a(\xi) \right] \in H^*(\bar{d}, L(\xi)). \quad (2.21)$$

Remarks:

(i) *Relation of the construction to Lie-Massey brackets:* For all $r \geq 2$, the constants $f_{a_1 \dots a_r}^b$ define higher order maps l_r in cohomology,

$$l_r : \otimes^r H(d) \longrightarrow H(d), \\ l_r([e_{a_1}], \dots, [e_{a_r}]) = f_{a_1 \dots a_r}^b [e_b]. \quad (2.22)$$

For a given $r \geq 2$, let us suppose that the $[e_{a_1}], \dots, [e_{a_r}]$ are such that the structure constants with strictly less than r indices vanish, for all choices of a_i 's. From the explicit form of the identity (2.16), it then follows that

$$\frac{1}{2} \sum_{k=1}^{r-1} (e_{(a_1 \dots a_k, e_{a_{k+1} \dots a_r})) (-)^{(k(1+\epsilon) + |e_{a_1}| + \dots + |e_{a_k}|)((r-k+1)(1+\epsilon) + \epsilon + |e_{a_{k+1}}| + \dots + |e_{a_{k+1}}|)} \\ + de_{a_1 \dots a_r} + e_b f_{a_1 \dots a_r}^b = 0, \quad (2.23)$$

the round bracket for the indices in the first term denoting symmetrization with respect to the grading of the ξ^{a_i} 's. We thus see that, under the above assumption,

$$-l_r([e_{a_1}], \dots, [e_{a_r}]) = \\ \frac{1}{2} \sum_{k=1}^{r-1} [(e_{(a_1 \dots a_k, e_{a_{k+1} \dots a_r})) (-)^{(k(1+\epsilon) + |e_{a_1}| + \dots + |e_{a_k}|)((r-k+1)(1+\epsilon) + \epsilon + |e_{a_{k+1}}| + \dots + |e_{a_{k+1}}|)}]. \quad (2.24)$$

By comparing with the definitions in [67], we identify the maps l_r , under the above assumption, as the value of the r -place Lie-Massey bracket $[[e_{a_1}], \dots, [e_{a_r}]]_M$ for the defining system $\{e_{a_{i_1} \dots a_{i_k}}, k = 1, \dots, r-1, 1 \leq i_1 < \dots < i_k \leq r\}$, up to a sign.

(ii) The differential Δ encodes a strongly homotopy Lie algebra on K [68].

(iii) *Iteration of the construction:* Because $(RDer[\wedge(sK^*)], [\cdot, \cdot]_C, d_\Delta)$ is again a graded differential Lie algebra, the main theorem can be applied to this graded differential Lie algebra, yielding a new graded differential Lie algebra, to which the main theorem can be applied, ...

2.3 Proof of the main theorem

2.3.1 Auxiliary acyclic extended graded differential Lie algebra

Let us now consider the graded differential Lie algebra $RDer[\wedge(sK^*)]^\epsilon$, which is obtained from the even graded Lie algebra $RDer[\wedge(sK^*)]$ by taking as new grading

of all the elements the old one minus ϵ . Explicitly, we introduce additional variables ξ_a^* of degree $-tot(\xi^a) - \epsilon = |e_a| - 1$ replacing the $\frac{\partial^R}{\partial \xi^a}$, and the space $RDer[\wedge(sK^*)]^\epsilon$ is composed of elements of the form $\xi_b^* \lambda^b(\xi)$. The even commutator bracket $[\cdot, \cdot]_C$ is replaced by

$$[\cdot, \cdot]_\xi : (RDer[\wedge(sK^*)]^\epsilon)^{g_1} \otimes (RDer[\wedge(sK^*)]^\epsilon)^{g_2} \longrightarrow (RDer[\wedge(sK^*)]^\epsilon)^{g_1+g_2+\epsilon},$$

$$[\lambda, \mu]_\xi = \frac{\partial^R \lambda}{\partial \xi^a} \frac{\partial^L \mu}{\partial \xi_a^*} - (-)^{a(a+\epsilon)} \frac{\partial^R \mu}{\partial \xi_a^*} \frac{\partial^L \lambda}{\partial \xi^a}, \quad (2.25)$$

where the shorthand notation a has been used for the grading of ξ^a , and $\lambda, \mu \in RDer[\wedge(sK^*)]^\epsilon$. It follows that $RDer[\wedge(sK^*)]^\epsilon$ is a graded Lie algebra with the same ϵ as the original one.

The auxiliary extended space is then defined to be

$$L(\xi, \xi^*) \equiv L \otimes \wedge(sK^*) \oplus RDer[\wedge(sK^*)]^\epsilon. \quad (2.26)$$

The differential d and the bracket $[\cdot, \cdot]$ on $L(\xi) = L \otimes \wedge(sK^*)$ are extended trivially to $L(\xi, \xi^*)$, while the bracket $[\cdot, \cdot]_\xi$ is extended by the formula (2.25), with λ, μ replaced by elements $A, B \in L(\xi, \xi^*)$. This means in particular that $RDer[\wedge(sK^*)]^\epsilon$ acts on $L(\xi)$ through the bracket $[\cdot, \cdot]_\xi$. The bracket

$$[\cdot, \cdot]^\sim = [\cdot, \cdot] + [\cdot, \cdot]_\xi \quad (2.27)$$

is then a graded Lie bracket of degree ϵ and the auxiliary differential graded Lie algebra is $(L(\xi, \xi^*), d, [\cdot, \cdot]^\sim)$.

In the general case where the differential d is not inner, a further extension is needed: define an element \bar{D} of degree $1 - \epsilon$ and consider the direct sum of the one dimensional vector space generated by \bar{D} with $L(\xi, \xi^*)$, $\bar{L}(\xi, \xi^*) = k\bar{D} \oplus L(\xi, \xi^*)$. The space $\bar{L}(\xi, \xi^*)$ is turned into a graded Lie algebra by extending the bracket $[\cdot, \cdot]^\sim$ according to

$$\frac{1}{2}[\bar{D}, \bar{D}]^\sim = 0,$$

$$[\bar{D}, \xi^{a_1} \dots \xi^{a_m} x]^\sim = s(\xi^{a_1} \dots \xi^{a_m} x) = -(-)^{tot(\xi^{a_1} \dots \xi^{a_m} x) + \epsilon} [\xi^{a_1} \dots \xi^{a_m} x, \bar{D}]^\sim,$$

$$[\bar{D}, \xi_b^* \lambda^b(\xi)]^\sim = 0 = [\xi_b^* \lambda^b(\xi), \bar{D}]^\sim = 0. \quad (2.28)$$

It follows that $(\bar{L}(\xi, \xi^*), d = [\bar{D}, \cdot]^\sim, [\cdot, \cdot]^\sim)$ is a graded differential Lie algebra with an inner differential.

In the case where the differential d on L is inner, one does not need to further extend the space, $\bar{L}(\xi, \xi^*) = L(\xi, \xi^*)$ and $\bar{D} = D \in L(\xi, \xi^*)$.

Define now

$$\bar{D}^1 = \bar{D} + e_a \xi^a \in \bar{L}^{1-\epsilon}(\xi, \xi^*) \quad (2.29)$$

and

$$\tilde{d} = d + e_a \frac{\partial^L}{\partial \xi_a^*} = [\bar{D}^1, \cdot]^\sim + [e_a \xi^a, \cdot]_\xi. \quad (2.30)$$

It follows directly that $\tilde{\delta}^2 = 0$ in $L(\xi, \xi^*)$, but also that the cohomology of $\tilde{\delta}$ is trivial,

$$H(\tilde{\delta}, L(\xi, \xi^*)) = 0. \quad (2.31)$$

Indeed, the condition $\tilde{\delta}[\xi^{a_1} \dots \xi^{a_m} x + \xi_a^* \lambda^a(\xi)] = 0$ implies separately $dx = 0 \Leftrightarrow x = e_b \mu^b + dy$ and $\lambda^a(\xi) = 0$, so that every cocycle is a coboundary, $\xi^{a_1} \dots \xi^{a_m} x + \xi_a^* \lambda^a(\xi) = \xi^{a_1} \dots \xi^{a_m} x = \tilde{\delta}[(-)^{\text{tot}(\xi^{a_1} \dots \xi^{a_m})} \xi^{a_1} \dots \xi^{a_m} (y + \xi_b^* \mu^b)]$.

2.3.2 Master equation in $\bar{L}(\xi, \xi^*)$ with acyclic differential through HPT

Take as resolution degree in $L(\xi, \xi^*)$ the number of ξ 's and define $\tilde{d}^1 = [\bar{D}^1, \cdot] = \tilde{\delta} + [e_a \xi^a, \cdot]$. The bracket $\frac{1}{2}[\bar{D}^1, \bar{D}^1] = \frac{1}{2}[e_a \xi^a, e_b \xi^b]$ is of resolution degree 2, so that $(\tilde{d}^1)^2 = \frac{1}{2}[[e_a \xi^a, e_b \xi^b], \cdot]$ is of resolution degree 1.

Suppose that we have constructed $\bar{D}^k = \bar{D} + \bar{D}_1 + \dots \bar{D}_k$, with $\bar{D}_i \in \bar{L}^{1-\epsilon}(\xi, \xi^*)$ for $i = 1, \dots, k$ of resolution degree i such that $\frac{1}{2}[\bar{D}^k, \bar{D}^k] = R_{k+1} + R_{r>k+1}$, where $R_{k+1}, R_{r>k+1} \in L(\xi, \xi^*)^{2-\epsilon}$ have respectively resolution degree $k+1$ and $r > k+1$. From $0 = [\bar{D}^k, \frac{1}{2}[\bar{D}^k, \bar{D}^k]] = \tilde{\delta}R_{k+1} + R'_{r>k+1}$, we find at resolution degree $k+1$ that $\tilde{\delta}R_{k+1} = 0$ and hence that $R_{k+1} = -\tilde{\delta}\bar{D}_{k+1}$, where $\bar{D}_{k+1} \in L(\xi, \xi^*)^{1-\epsilon}$ is of resolution degree $k+1$. It follows that if $\bar{D}^{k+1} = \bar{D}^k + \bar{D}_{k+1}$, $\frac{1}{2}[\bar{D}^{k+1}, \bar{D}^{k+1}] = R''_{r>k+1}$. Hence, the construction can be continued recursively to get a $\bar{D}(\xi, \xi^*) = \bar{D} + \sum_{k \geq 1} \bar{D}_k \in \bar{L}(\xi, \xi^*)^{1-\epsilon}$ of the form

$$\bar{D}(\xi, \xi^*) = \bar{D} + e(\xi) + \xi_b^* f^b(\xi), \quad (2.32)$$

such that

$$\frac{1}{2}[\bar{D}(\xi, \xi^*), \bar{D}(\xi, \xi^*)] = 0. \quad (2.33)$$

The corresponding differential in $L(\xi, \xi^*)$ is

$$\tilde{d} = [\bar{D}(\xi, \xi^*), \cdot]. \quad (2.34)$$

The cohomology of \tilde{d} is trivial,

$$H^*(\tilde{d}, L(\xi, \xi^*)) = 0. \quad (2.35)$$

Indeed, developing the cocycle l in $\tilde{d}l = 0$ according to the resolution degree, $l = \sum_{r \geq M} l_r$ we get at lowest order $\tilde{d}l_M = 0 \Rightarrow l_M = \tilde{\delta}k_M$, so that $l - \tilde{d}k_M$ starts at resolution degree $M+1$ and one can continue recursively to show that $l = \tilde{d}k$ for some $k \in L(\xi, \xi^*)$.

2.3.3 Decomposition

If $\bar{V}(\xi) = k\bar{D} \oplus V(\xi)$, the solution $\bar{D}(\xi, \xi^*)$ splits into

$$\begin{aligned} D(\xi) &= \bar{D} + e(\xi) \in \bar{V}(\xi), \\ \Delta^* &= \xi_b^* f^b(\xi) \in RDer[\wedge(sK^*)]^\epsilon, \end{aligned} \quad (2.36)$$

The master equation (2.33) in $RDer[\wedge(sK^*)]^\epsilon$ gives

$$\frac{1}{2}[\Delta^*, \Delta^*]_\xi = 0. \quad (2.37)$$

This equation is equivalent to $\Delta^2 = 0$ and proves (2.15). Explicitly, for each $r \geq 3$, it gives the set of higher order Jacobi identities

$$\sum_{m=2}^{r-1} m f_{(a_1 \dots a_{m-1} | b|}^c f_{a_m \dots a_r}^b = 0, \quad (2.38)$$

where the round parenthesis denote graded symmetrization with respect to the grading of the ξ^a .

The associated differential in $RDer[\wedge(sK^*)]^\epsilon$ is $d_{\Delta^*} = [\Delta^*, \cdot]_\xi$. The graded differential Lie algebra $(RDer[\wedge(sK^*)]^\epsilon, d_{\Delta^*}, [\cdot, \cdot]_\xi)$ can be transformed to the even graded differential Lie algebra $(RDer[\wedge(sK^*)], d_\Delta, [\cdot, \cdot]_C)$ through the substitution $\xi_b^* \leftrightarrow \frac{\partial^R}{\partial \xi^b}$. The master equation (2.33) in $V(\xi)$ reduces to equation (2.16), while in the case of an inner differential $d = [D, \cdot]$, it reduces to the quantum master equation (2.19).

The isomorphism between $H^*(\bar{d}, V(\xi))$ and $H^*(d_\Delta, RDer[\wedge(sK^*)])$ follows directly from the triviality of the cohomology of \tilde{d} . Indeed, for $l = a(\xi) + \xi_a^* \nu^a(\xi)$ and $k = b(\xi) + \xi_a^* \lambda^a(\xi)$, the relation $\tilde{d}l = 0 \iff l = \tilde{d}k$ gives, in the case where $\nu^a(\xi) = 0$,

$$\bar{d}a(\xi) = 0 \iff \begin{cases} a(\xi) = \bar{d}b(\xi) + \frac{\partial^R e(\xi)}{\partial \xi^a} \lambda^a(\xi), \\ [\Delta^*, \xi_b^* \lambda^b(\xi)]_\xi = 0. \end{cases} \quad (2.39)$$

In order to find the cohomology, we need to know when this decomposition is direct, i.e., we need to solve $\bar{d}b(\xi) + \frac{\partial^R e(\xi)}{\partial \xi^a} \lambda^a(\xi) = 0$, under the condition $[\Delta^*, \xi_b^* \lambda^b(\xi)]_\xi = 0$. This is equivalent to $\tilde{d}(b(\xi) + \xi_a^* \lambda^a(\xi)) = 0$, and, using again the triviality of the cohomology of \tilde{d} , this implies that $b(\xi) + \xi_a^* \lambda^a(\xi) = \tilde{d}(c(\xi) + \xi_c^* \mu^c(\xi))$, or explicitly,

$$\begin{cases} \bar{d}b(\xi) + \frac{\partial^R e(\xi)}{\partial \xi^a} \lambda^a(\xi) = 0, \\ [\Delta^*, \xi_b^* \lambda^b(\xi)]_\xi = 0, \end{cases} \iff \begin{cases} b(\xi) = \bar{d}c(\xi) + \frac{\partial^R e(\xi)}{\partial \xi^a} \mu^a(\xi), \\ \xi_a^* \lambda^a(\xi) = [\Delta^*, \xi_c^* \mu^c(\xi)]_\xi, \end{cases} \quad (2.40)$$

This implies that the decomposition is direct iff $\xi_a^* \lambda^a(\xi)$ is not a coboundary and proves (2.21).

2.4 Spectral sequence for the computation of the cohomologies of d_Δ and \bar{d}

The purpose of this subsection is to discuss briefly the general computation of the cohomology $H(d_\Delta, RDer[\wedge(sK^*)])$ (and thus also of $H(\bar{d}, L(\xi))$ because of the isomorphism (2.20), (2.21)). In order to do so, we give some details on exact couples and spectral sequences and apply these concepts to the present problem.

Let $\lambda = \frac{\partial^R}{\partial \xi^A} \lambda^A$ be a right derivation. We assume that the λ^A are formal power series in ξ^A . In the following, we provide this space with an obvious filtration. It will however not have finite length in general, and for particular theories, better filtrations have to be found in order to do a complete computation. Since the techniques will be similar, it is nevertheless useful to show how they work for this filtration.

2.4.1 Grading and filtration on the space of right derivations

Let $N_\xi = \frac{\partial^R}{\partial \xi^A} \xi^A$ be the operator counting the number of ξ 's. A general right derivation admits the following decomposition according to the eigenvalues of N_ξ : $\lambda = \lambda_{-1} + \lambda_0 + \lambda_1 + \dots$, where $[\lambda_p, N_\xi] = p\lambda_p$. Hence, $RDer[\wedge(sK^*)]$ is a graded space, $RDer[\wedge(sK^*)] = \oplus_{p=-1} RDer[\wedge(sK^*)]^p$. (It is actually a bigraded space, the other grading, for which d_Δ is homogeneous of degree 1 being the grading *tot.*)

The graded right commutator satisfies $[[\lambda_m, \mu_n], N_\xi] = (m+n)[\lambda_m, \mu_n]$. The decomposition of Δ starts at eigenvalue 1: $\Delta = \Delta_1 + \Delta_2 + \dots$; the corresponding decomposition of d_Δ being $d_\Delta = [\Delta_1, \cdot] + [\Delta_2, \cdot] + \dots \equiv d_1 + d_2 + \dots$. It follows that the cocycle condition $d_\Delta \lambda = 0$ decomposes as

$$\begin{aligned} d_1 \lambda_{-1} &= 0, \\ d_1 \lambda_0 + d_2 \lambda_{-1} &= 0, \\ d_1 \lambda_1 + d_2 \lambda_0 + d_3 \lambda_{-1} &= 0, \\ &\vdots, \end{aligned} \tag{2.41}$$

while the coboundary condition $\lambda = d_\Delta \mu$ decomposes as

$$\begin{aligned} \lambda_{-1} &= 0, \\ \lambda_0 &= d_1 \mu_{-1}, \\ \lambda_1 &= d_1 \mu_0 + d_2 \mu_{-1}, \\ \lambda_2 &= d_1 \mu_1 + d_2 \mu_0 + d_3 \mu_{-1}, \\ &\vdots, \end{aligned} \tag{2.42}$$

In order to construct the spectral sequence associated to this problem, we follow [69].

Consider the spaces K_p of derivations having N_ξ degree greater than p , i.e., $\lambda \in K_p$ if $\lambda = \lambda_p + \lambda_{p+1} + \dots$. The space of all right derivations is $RDer[\wedge(sK^*)] = K_{-1}$, $K_{p+1} \subset K_p$ and $d_\Delta K_p \subset K_p$. The sequence of spaces K_p is a decreasing filtration of $RDer[\wedge(sK^*)]$, with $K_p/K_{p+1} \simeq RDer[\wedge(sK^*)]^p$.

We have the short exact sequence⁴:

$$0 \longrightarrow \oplus_{p=-1} K_{p+1} \xrightarrow{i} \oplus_{p=-1} K_p \xrightarrow{j} \oplus_{p=-1} K_p/K_{p+1} \longrightarrow 0, \tag{2.43}$$

where $\oplus_{p=-1} K_p/K_{p+1} \simeq \oplus_{p=-1} RDer[\wedge(sK^*)]^p$. The following diagram is exact at each corner:

$$\begin{array}{ccc} H(d_\Delta, \oplus_{p=-1} K_{p+1}) & \xrightarrow{i_0} & H(d_\Delta, \oplus_{p=-1} K_p) \\ & \searrow k_0 \quad \swarrow j_0 & \\ & E_0, & \end{array} \tag{2.44}$$

⁴A diagram is said to be exact if the image of a map is equal to the kernel of the next map.

where $E_0 = \oplus_{p=-1} K_p / K_{p+1} \simeq \oplus_{p=-1} RDer[\wedge(sK^*)]^p$. In this diagram, $H(d_\Delta, K_p)$ is defined by the cocycle condition $d_\Delta(\lambda_p + \lambda_{p+1} + \dots) = 0$, and the coboundary condition $\lambda_p + \lambda_{p+1} + \dots = d_\Delta(\mu_p + \mu_{p+1} + \dots)$. The maps i_0 and j_0 are induced by i and j , $i_0[\lambda_{p+1} + \lambda_{p+2} + \dots] = [\lambda_{p+1} + \lambda_{p+2} + \dots]$ and $j_0[\lambda_p + \lambda_{p+1} + \dots] = [j(\lambda_p + \lambda_{p+1} + \dots)] = [\lambda_p]$. They are well defined, because i_0 maps cocycles to cocycles and coboundaries to coboundaries, while $j(d_\Delta(\mu_p + \mu_{p+1} + \dots)) \in K_{p+1}$. The map k_0 is defined by $k_0[\lambda_p] = [d_\Delta \lambda_p]$. It does not depend on the choice of representative for $[\lambda_p] \in K_p / K_{p+1}$, because $[d_\Delta(\lambda_{p+1} + \dots)] = 0 \in H(d_\Delta, K_{p+1})$.

Let us check explicitly that this diagram is exact:

- $\ker j_0$ is given by elements $[\lambda_p + \lambda_{p+1} + \dots] \in H(d_\Delta, K_p)$ such that $d_\Delta(\lambda_p + \lambda_{p+1} + \dots) = 0$ and $\lambda_p = 0$. This is the same than $i_0 H(s, K_{p+1})$, which is given by $[\lambda_{p+1} + \lambda_{p+2} + \dots]$, with $d_\Delta(\lambda_{p+1} + \lambda_{p+2} + \dots) = 0$, the equivalence relation being the equivalence relation in $H(s, K_p)$ by definition of i_0 .
- $\ker k_0$ is given by elements $[\lambda_p]$ such that $[d_\Delta \lambda_p] = 0 \in H(d_\Delta, K_{p+1})$, i.e. such that $d_\Delta \lambda_p = d_\Delta(\mu_{p+1} + \mu_{p+2} + \dots)$. By the identification $\lambda_{p+1} = -\mu_{p+1}$, $\lambda_{p+2} = -\mu_{p+2}$, \dots , this is indeed the same than $j_0 H(d_\Delta, K_p)$ given by $[\lambda_p]$ with $d_\Delta(\lambda_p + \lambda_{p+1} + \dots) = 0$.
- $\ker i_0$ is given by elements $[\lambda_{p+1} + \lambda_{p+2} + \dots]$ such that $d_\Delta(\lambda_{p+1} + \lambda_{p+2} + \dots) = 0$ and $\lambda_{p+1} + \lambda_{p+2} + \dots = d_\Delta(\mu_p + \mu_{p+1} + \dots)$, while $k_0[\mu_p]$ is given by $[\lambda_{p+1} + \lambda_{p+2} + \dots]$ of the form $[d_\Delta \mu_p]$ so that $\lambda_{p+1} + \lambda_{p+2} + \dots = d_\Delta \mu_p + d_\Delta(\mu_{p+1} + \dots)$, which is indeed the same.

2.4.2 Exact couples and associated spectral sequence

To every exact couple (A_0, B_0) , i.e., exact diagram of the form

$$\begin{array}{ccc} A_0 & \xrightarrow{i_0} & A_0 \\ k_0 \swarrow & & \searrow j_0 \\ & B_0, & \end{array} \quad (2.45)$$

one can associated a derived exact couple

$$\begin{array}{ccc} A_1 & \xrightarrow{i_1} & A_1 \\ k_1 \swarrow & & \searrow j_1 \\ & B_1. & \end{array} \quad (2.46)$$

In this diagram, the spaces and maps are defined as follows: $A_1 = i_0 A_0$; $B_1 = H(d_0, B_0)$, where $d_0 = j_0 \circ k_0$ ($d_0^2 = 0$ because $k_0 \circ j_0 = 0$); for $a_1 = i_0 a_0$, $i_1 a_1 = i_1(i_0 a_0) = i_0^2 a_0$; $j_1 a_1 = [j_0 a_0]$ (this map is well defined: $j_0 a_0$ is a cocycle, because $k_0 \circ j_0 = 0$, furthermore the map does not depend on the representative chooser for a_0 , because if $i_0 a_0 = 0$, $a_0 = k_0 b_0$ for some b_0 and $j_1 a_1 = [j_0 \circ k_0 b_0] = 0$); $k_1[b_0] = k_0 b_0$ ($k_0 b_0 = i_0 a_0$ for some b_0 because $d_0 b_0 = j_0(k_0 b_0) = 0$ implies $k_0 b_0 = i_0 a_0$, furthermore $k_0 d_0 c_0 = 0$ because $k_0 \circ j_0 = 0$).

Let us also check explicitly exactness of this diagram:

- $\ker j_1$ is given by elements $a_1 = i_0 a_0$ such that $[j_0 a_0] = 0$, i.e., $j_0 a_0 = j_0 k_0 b_0$ and then $a_0 - k_0 b_0 = i_0 c_0$, implying that $a_1 = i_0^2 c_0$. $i_1 A_1$ is given by elements $a_1 = i_1 c_1 = i_0^2 c_0$. It follows that $\ker j_1 \subset i_1 A_1$, while the inverse inclusion follows from $j_0 \circ i_0 = 0$.
- $\ker k_1$ is given by elements $[b_0]$ such that $k_0 b_0 = i_0 a_0 = 0$, i.e., such that $b_0 = j_0 c_0$, for some c_0 , while $\text{im } j_1$ is given by elements $[b_0]$ such that $[b_0] = [j_0 e_0]$, i.e. $b_0 = j_0(e_0 + k_0 f_0)$. It follows that $\ker k_1 = \text{im } j_1$.
- $\ker i_1$ is given by elements $a_1 = i_0 a_0$ such that $i_0(i_0 a_0) = 0$, i.e., $i_0 a_0 = k_0 b_0$ (which implies in particular $d_0 b_0 = 0$). $\text{im } k_1$ is given by elements $a_1 = i_0 a_0 = k_0 b_0$ for some b_0 with $d_0 b_0 = 0$, so both spaces are indeed the same.

Clearly, this construction can be iterated by taking as the starting exact couple the derived couple. We thus get a sequence of exact couples

$$\begin{array}{ccc} A_r & \xrightarrow{i_r} & A_r \\ k_r \swarrow & & \searrow j_r \\ & B_r & \end{array} \quad (2.47)$$

and the associated spectral sequence (B_r, d_r) , for $r = 0, 1, \dots$, i.e., spaces B_r and differentials d_r satisfying $B_{r+1} = H(d_r, B_r)$.

2.4.3 Spectral sequence associated to d_Δ

Let us now apply the general theory to the case of the exact couple (2.44) and give explicitly the differentials d_r and the spaces B_r (called E_r) in this case for $r = 0, 1, 2, 3$.

We have $E_0 = \oplus_{p=-1} K_p / K_{p+1} \simeq \oplus_{p=-1} R\text{Der}[\wedge(sK^*)]^p$. The differential d_0 is defined by $d_0[\lambda_p]_0 = j_0[d_\Delta \lambda_p]$, where $[d_\Delta \lambda_p] \in H(d_\Delta, K_{p+1})$. It follows that $d_0[\lambda_p]_0 = [d_1 \lambda_p]$. This means that E_1^p is defined by elements $[[\lambda_p]_0]_1$ with the cocycle condition

$$d_1 \lambda_p = 0 \quad (2.48)$$

and the coboundary condition

$$\lambda_p = d_1 \mu_{p-1}. \quad (2.49)$$

Because $d_1 = \frac{\partial^R}{\partial \xi^a} f_{bc}^a \xi^b \xi^c$, and $d_1^2 = 0$ implies that the f_{bc}^a are the structure constants of a graded Lie algebra, this group is just a graded version of standard Lie algebra (Chevalley-Eilenberg) cohomology with representation space the adjoint representation.

Take now $[[\lambda_p]_0]_1 \in E_1^p$. The differential $d_1[[\lambda_p]_0]_1 = j_1 k_1[[\lambda_p]_0]_1 = j_1 k_0[\lambda_p]_0 = j_1[d_\Delta \lambda_p] = [j_0 i_0^{-1}[d_\Delta \lambda_p]]_1$. This means that $[d_\Delta \lambda_p]$ has to be considered as an element of $H(d_\Delta, K_{p+2})$ so that $d_1[[\lambda_p]_0]_1 = [[d_2 \lambda_p]_0]_1$. Hence E_2^p is defined by elements $[[[\lambda_p]_0]_1]_2$ with the cocycle condition

$$d_2 \lambda_p + d_1 \lambda_{p+1} = 0, \quad (2.50)$$

$$d_1 \lambda_p = 0, \quad (2.51)$$

and the coboundary condition

$$\lambda_p = d_2\mu_{p-2} + d_1\mu_{p-1}, \quad (2.52)$$

$$0 = d_1\mu_{p-2}. \quad (2.53)$$

We thus find that $E_2^p = H^p(d_2, H(d_1))$.

The differential d_2 in E_2^p is defined by $d_2[[[\lambda_p]_0]_1]_2 = j_2k_2[[[\lambda_p]_0]_1]_2 = j_2k_1[[[\lambda_p]_0]_1]_2 = j_2k_0[\lambda_p]_0 = [j_1i_1^{-1}k_0[\lambda_p]_0]_2 = [[j_0i_0^{-1}i_1^{-1}k_0[\lambda_p]_0]_1]_2$. In order to make sure that $k_0[\lambda_p]_0$ belongs to $i_1i_0H(d_\Delta, K_{p+1})$ we use $\lambda_p + \lambda_{p+1}$ as a representative for $[\lambda_p]_0$. It follows that $d_2[[[\lambda_p]_0]_1]_2 = [[[d_3\lambda_p + d_2\lambda_{p+1}]_0]_1]_2$. The cocycle condition for an element $[[[[[\lambda_p]_0]_1]_2]_3] \in E_3^p$ is then given by

$$d_3\lambda_p + d_2\lambda_{p+1} = d_2\mu_{p+1} + d_1\mu_{p+2}, \quad (2.54)$$

$$d_2\lambda_p + d_1\lambda_{p+1} = 0, \quad (2.55)$$

$$d_1\lambda_p = 0, \quad (2.56)$$

with $d_1\mu_{p+2} = 0$. The redefinition $\lambda_{p+1} \rightarrow \lambda_{p+1} - \mu_{p+1}$ and $\lambda_{p+2} = -\mu_{p+2}$, then gives as cocycle condition

$$d_3\lambda_p + d_2\lambda_{p+1} + d_1\lambda_{p+2} = 0, \quad (2.57)$$

$$d_2\lambda_p + d_1\lambda_{p+1} = 0, \quad (2.58)$$

$$d_1\lambda_p = 0. \quad (2.59)$$

The coboundary condition is $[[[\lambda_p]_0]_1]_2 = d_3[[[\mu_{p-3} + \mu_{p-2}]_0]_1]_2$, where $d_1\mu_{p-3} = 0, d_2\mu_{p-3} + d_2\mu_{p-2} = 0$, hence $[[[\lambda_p]_0]_1]_2 = [[[d_3\mu_{p-3} + d_2\mu_{p-2}]_0]_1]_2 + d_2[[[\sigma_{p-2}]_0]_1]_2$, with $d_1\sigma_{p-2} = 0$ which gives

$$\lambda_p = d_3\mu_{p-3} + d_2\mu_{p-2} + d_2\sigma_{p-2} + d_1\rho_{p-1}, \quad (2.60)$$

$$0 = d_2\mu_{p-3} + d_1\mu_{p-2}, \quad (2.61)$$

$$0 = d_1\mu_{p-3}, \quad 0 = d_1\sigma_{p-2}. \quad (2.62)$$

The redefinition $\mu_{p-2} \rightarrow \mu_{p-2} + \sigma_{p-2}$ and $\rho_{p-1} = \mu_{p-1}$, then gives the coboundary condition

$$\lambda_p = d_3\mu_{p-3} + d_2\mu_{p-2} + d_1\mu_{p-1}, \quad (2.63)$$

$$0 = d_2\mu_{p-3} + d_1\mu_{p-2}, \quad (2.64)$$

$$0 = d_1\mu_{p-3}. \quad (2.65)$$

This construction can be continued in the same way for higher r 's.

The original problem was the computation of $H(d_\Delta, RDer[\wedge(sK^*)]) = H(d_\Delta, K_{-1})$. From exactness of the couples (2.47), it follows that

$$\begin{aligned} H(d_\Delta, K_{-1}) &\simeq j_0E_0^{-1} \oplus \ker j_0 \\ &\simeq \ker k_0(\subset E_0^{-1}) \oplus i_0H(d_\Delta, K_0) \\ &\simeq \ker k_0(\subset E_0^{-1}) \oplus \ker k_1(\subset E_1^0) \oplus i_1i_0H(d_\Delta, K_1) \end{aligned} \quad (2.66)$$

$$\vdots \quad (2.67)$$

$$\simeq \oplus_{r=0}^R \ker k_r(\subset E_r^{r-1}) \oplus i_R \dots i_0H(d_\Delta, K_R). \quad (2.68)$$

Furthermore, $E_0 \simeq E_1 \oplus F_0 \oplus d_0 F_0$ and $E_0^{-1} \simeq E_1^{-1} \oplus F_0^{-1}$. F_0 does not belong to $\ker k_0$ because $d_0 F_0 \neq 0$. Thus $\ker k_0(\subset E_0^{-1}) \simeq \ker k_1(\subset E_1^{-1})$. Similarly, $E_1 \simeq E_2 \oplus F_1 \oplus d_1 F_1$ and $(d_1 F_1)^{-1} = (d_1 F_1)^0 = 0$. Again, $d_1[F_1]_1 \neq 0$ implies that F_1 does not belong to $\ker k_1$. This means that $\ker k_1(\subset E_1^{-1}) \simeq \ker k_2(\subset E_2^{-1})$ and $\ker k_1(\subset E_1^0) \simeq \ker k_2(\subset E_2^0)$. Going on in the same way, we conclude that $\ker k_r(\subset E_r^{-1}) \simeq \ker k_R(\subset E_R^{-1})$. We thus get

$$H(d_\Delta, K_{-1}) \simeq \oplus_{r=0}^R \ker k_R(\subset E_R^{-1}) \oplus i_R \dots i_0 H(d_\Delta, K_R). \quad (2.69)$$

This construction is most useful if it would stop at some point. Indeed, suppose that $K_R = 0$. Because $k_R[\dots[\lambda_p]_0 \dots]_R$ belongs to $i_R \dots i_0 H(d_\Delta, K_{p+R+1}) = 0$, it follows that

$$H(d_\Delta, K_{-1}) \simeq \oplus_{r=0}^{R-1} E_R^{-1}. \quad (2.70)$$

2.5 Examples

2.5.1 Lie algebra cochains and chains

As a first example, we consider the tensor product of chains and cochains on a semi-simple Lie algebra \mathcal{G} with generators T_I . This space can be identified with the space

$$L = \wedge(\mathcal{G}) \otimes \wedge(\mathcal{G}^*) = \wedge(\mathcal{P}_I, C^I), \quad (2.71)$$

where \mathcal{P}_I and C^I are Grassmann odd variables. The grading is obtained by giving degree -1 to \mathcal{P}_I and degree 1 to C^I . The graded Lie bracket on L (with $\epsilon = 0$) is taken to be

$$\{\cdot, \cdot\} = \frac{\partial^R}{\partial \mathcal{P}_I} \frac{\partial^L}{\partial C^I} + \frac{\partial^R}{\partial C^I} \frac{\partial^L}{\partial \mathcal{P}_I}. \quad (2.72)$$

The differential d on L is inner,

$$\Omega = \frac{1}{2} f_{IJ}{}^K \mathcal{P}_K C^J C^I, \quad d = \{\Omega, \cdot\}, \quad (2.73)$$

$$d = C^J f_{JI}{}^K \mathcal{P}_K \frac{\partial^L}{\partial \mathcal{P}_I} + \frac{1}{2} C^J C^I f_{IJ}{}^K \frac{\partial^L}{\partial C^K}. \quad (2.74)$$

It is the standard Chevalley-Eilenberg differential with representation space the space of chains transforming under the extension of the adjoint representation. The interesting feature of this example is that the cohomology $H(d, L)$ can be computed exactly: it is generated by the primitive invariant chains and the primitive invariant cochains, which are completely known for semi-simple Lie algebras (see e.g. [72]).

Note that this example has more structure than discussed in the general case. Indeed, $(L, d, \{\cdot, \cdot\})$ is a graded differential Poisson algebra for the exterior product of chains and cochains.

In the particular case of $su(2)$ a basis $\{e_a\}$ of $K \simeq H(d, L)$ is given by

$$e_1 = 1, e_2 = \frac{1}{3} f_{IJK} C^I C^J C^K, e_3 = \frac{1}{3} f^{IJK} \mathcal{P}_I \mathcal{P}_J \mathcal{P}_K, e_4 = e_2 e_3, \quad (2.75)$$

where the indices are lowered and raised with the Killing metric $g_{IJ} = f_{IL}{}^K f_{JK}{}^L$ and its inverse. Direct computation shows that only one bracket of the elements of the basis is non vanishing and that it is d exact, $\{\xi^2 e_2, \xi^3 e_3\} + \{\Omega, \xi^2 \xi^3 e_{23}\} = 0$, where $e_{23} = -\frac{2}{3} f^{IJ}{}_K \mathcal{P}_I \mathcal{P}_J C^K$. Furthermore, $\{e_{23}, e_a\} = 0 = \{e_{23}, e_{23}\}$. It follows that $\Delta = 0$ and that $\Omega(\xi) = \Omega + \xi^a e_a + \xi^2 \xi^3 e_{23}$ satisfies

$$\frac{1}{2} \{\Omega(\xi), \Omega(\xi)\} = 0. \quad (2.76)$$

In the case of a general semi-simple Lie algebra \mathcal{G} , it is not difficult to show that the first order structure functions f_{bc}^a all vanish. The computation of the higher order structure functions is more involved. The complete analysis of the sh Lie structure of the cohomology $H(d, L)$ and of the associated “quantum master equation” will be discussed elsewhere [70].

2.5.2 Schouten-Nijenhuis bracket in $S(\mathcal{G}) \otimes \wedge \mathcal{G}^*$

The second example is discussed in section 4 of [71]. The space L is taken to be the symmetric tensors on \mathcal{G} tensor product with the cochains,

$$L = S(\mathcal{G}) \otimes \wedge(\mathcal{G}^*) = \wedge(x_I, C^I), \quad (2.77)$$

where the x_I are generators of \mathcal{G} considered as even coordinates on \mathcal{G}^* , with degree 0, while the Grassmann odd generators C^I have degree 1. The Schouten-Nijenhuis bracket of grading $\epsilon = -1$ is defined by

$$(\cdot, \cdot) = \frac{\partial^R}{\partial x_I} \frac{\partial^L}{\partial C^I} - \frac{\partial^R}{\partial C^I} \frac{\partial^L}{\partial x_I}, \quad (2.78)$$

Again, the differential d on L is inner,

$$\Omega = \frac{1}{2} f_{IJ}{}^K x_K C^J C^I, \quad d = (\Omega, \cdot), \quad (2.79)$$

$$d = C^J f_{JI}{}^K x_K \frac{\partial^L}{\partial x_I} + \frac{1}{2} C^J C^I f_{IJ}{}^K \frac{\partial^L}{\partial C^K}. \quad (2.80)$$

It is the standard Chevalley-Eilenberg differential with representation space in this case the space of symmetric tensor on \mathcal{G} transforming under the extension of the adjoint representation. The cohomology $H(d, L)$ can also be computed exactly in this case: it is generated by primitive invariant symmetric tensors and primitive invariant cochains, which are again completely classified for semi-simple Lie algebras (see e.g. [72]).

In this example, L is a Gerstenhaber algebra: the Schouten-Nijenhuis bracket is a graded derivation of the exterior product. Furthermore, the bracket (\cdot, \cdot) measures the failure of the second order differential operator $\mathcal{D} = \frac{\partial^R}{\partial x_K} \frac{\partial^R}{\partial C^K}$ to be a derivation.

2.5.3 Hamiltonian BFV formalism

Another, physically relevant example of a differential graded Lie algebra is the formalism of Batalin, Fradkin and Vilkovisky for constrained Hamiltonian systems [73, 74, 75, 76, 77, 78] (for reviews, see e.g. [79, 80, 81, 82, 26]). The even graded Lie bracket $\{\cdot, \cdot\}$ is the extension of the standard Poisson bracket to the ghost and ghost momenta, while the inner differential is generated by the BRST charge Ω , built out of the constraints and the structure functions arising in their Poisson bracket algebra.

2.5.4 Lagrangian BV formalism

The general formalism to control gauge symmetries through a differential during quantization in the Lagrangian framework, developed by Becchi, Rouet, Stora [8, 9, 10, 11], Tyutin [12], Zinn-Justin [13, 14], Batalin and Vilkovisky [21, 22, 23, 24, 25] is referred to as the “antifield formalism” below. It is another physically important example of a differential graded Lie algebra. Since the main objective of this review is to analyze what can be gained by the general construction of section 2.2 in this context, the different ingredients of the antifield formalism will be briefly reviewed in the next section. More details can be found in the original papers, in the reviews [26, 83, 84, 27, 33] and in the references cited therein.

3 Antifield formalism and local BRST cohomology

3.1 Antibracket and master equation

The starting point for local gauge field theories is an action

$$S_0[\phi^i] = \int d^n x L_0, \quad (3.1)$$

where the Lagrangian L_0 depends on the fields ϕ^i and a finite number of their derivatives. The left hand side of the equations of motion are determined by the Euler-Lagrange derivatives of L_0 ,

$$\frac{\delta L_0}{\delta \phi^i} = 0. \quad (3.2)$$

In gauge theories, the left hand side of the equations of motion are not all independent, but there exist on-shell non vanishing relations among them. These are the non trivial Noether identities in one to one correspondence with the non trivial gauge symmetries.

The original set of fields $\{\phi^i\}$ is extended to the set $\{\phi^a\}$ by introducing in addition (i) ghost fields for the non trivial Noether identities, (ii) ghosts for ghosts associated to the non trivial reducibility identities of Noether identities, (iii) ghosts for ghosts for ghosts for the second stage non trivial reducibility identities. . . , and (iv) antifields ϕ_a^* associated to all of the above fields. In the following, we will denote the fields and antifields collectively by z^α .

In the classical theory, the relevant space is the space of local functionals \mathcal{F} in the fields and antifields. Under appropriate vanishing conditions on the fields, antifields and their derivatives at infinity, this space is isomorphic to the space of functions in the fields, the antifields and a finite number of their derivatives, up to total divergences. Introducing the horizontal one forms dx^μ and the horizontal differential $d = dx^\mu \partial_\mu$, with $\partial_\mu = \frac{\partial}{\partial x^\mu} + \sum_{k=0} z_{(\mu\nu_1 \dots \nu_k)}^\alpha \frac{\partial}{\partial z_{(\nu_1 \dots \nu_k)}^\alpha}$, the space of local functionals is isomorphic to the cohomology of the horizontal differential d in form degree n in the space Ω of form valued local functions, $\mathcal{F} \simeq H^n(d, \Omega)$. The grading is the ghost number obtained by assigning degree 0 to the original fields, degree 1 to the ghosts, 2 to the ghosts for ghosts. The ghost number of an antifield is defined to be minus the ghost number of the corresponding field minus 1. The odd graded Lie bracket with $\epsilon = 1$ of local functionals $A_i = \int d^n x a_i$ is defined by

$$(A_1, A_2) = \int d^n x \frac{\delta^R a_1}{\delta \phi^a} \frac{\delta^L a_2}{\delta \phi_a^*} - \frac{\delta^R a_1}{\delta \phi_a^*} \frac{\delta^L a_2}{\delta \phi^a}. \quad (3.3)$$

It is called “antibracket” in this context. In terms of generating functionals for Green’s functions introduced below, the antibracket is defined by a similar expression where the Euler-Lagrange derivatives of the integrands, $\frac{\delta a}{\delta z^\alpha}$ is replaced by the functional derivatives $\frac{\delta A}{\delta z^\alpha(x)}$ for the corresponding functionals. This generalization is consistent because for local functionals $\frac{\delta A}{\delta z^\alpha(x)} = \frac{\delta a}{\delta z^\alpha}(x)$.

The central object of the formalism is the construction of a solution S to the classical master equation

$$\frac{1}{2}(S, S) = 0. \quad (3.4)$$

This solution is obtained by first constructing a (possibly reducible) generating set of non trivial gauge symmetries, as well as generating sets of non trivial reducibility identities for the gauge symmetries, of non trivial reducibility identities of the second stage for the previous reducibility identities. . . .

If the solution S of the master equation (3.4) is required to be in ghost number 0, Grassmann even, minimal and proper, i.e., to contain in addition to the starting point action S_0 the gauge transformations related to the generating set of non trivial gauge symmetries as well as the various reducibility identities in a canonical way, one can show existence and locality of this solution, with uniqueness holding up to canonical field-antifield redefinitions. The BRST differential is then $s = (S, \cdot)$, so that $(\mathcal{F}, (\cdot, \cdot), s)$ is a graded differential Lie algebra with an inner differential.

3.2 Local BRST cohomology

The cohomology of s in the space of local functionals, $H^*(s, \mathcal{F})$ is called local BRST cohomology. With the algebraic characterization of local functionals discussed above, it is isomorphic to the cohomology of s modulo d in form degree n , $H^{*,n}(s|d, \Omega)$, in the space of form valued local functions Ω . This cohomological group is in turn related to the cohomology of s in Ω , $H(s, \Omega)$ through descent equations.

In negative ghost numbers g , the groups $H^{g,n}(s|d, \Omega)$ describe the generalized non trivial global symmetries of the theory. In ghost number zero, they describe the observables, i.e., the equivalence classes of local functionals that are gauge invariant when the gauge covariant equations of motion $\delta L_0 / \delta \phi^i = 0$ hold, where two such functionals, that coincide when these equations hold, have to be identified. The BRST cohomology in ghost number 0 also describes the infinitesimal deformations of the master equation, the obstructions to deformations being described by local BRST cohomological classes in ghost number 1. As will be discussed below, the local BRST cohomology in ghost number 0 also constrains the divergences and the counterterms of the quantum theory, while the classes in ghost number 1 describe the anomaly candidates.

3.3 Gauge fixing

In order to get well defined propagators, needed as a starting point for perturbation theory, the gauge has to be fixed. The gauge fixing can be done in two steps: first one adds a cohomological trivial non minimal sector. This amounts to extending the minimal solution of the master equation to $S' = S + \int d^n x B^a \bar{C}_a^*$. The canonical BRST differential extended to the antifields and the non minimal sector is $s = (S', \cdot)_{\phi, \phi^*}$. The second step is to perform an anticanonical transformation generated by a gauge fixing fermion $\Psi[\phi^a]$: the gauge fixed action to be used for quantization is $S_{\text{gf}}[\phi^a, \phi_a^*] =$

$S'[\phi^a, \tilde{\phi}_a^* + \frac{\delta^L \Psi}{\delta \phi^a}]$, with Ψ chosen in such a way that the propagators of the theory are well defined. For instance, in Yang-Mills type theories, standard linear gauges are obtained from

$$\Psi = \int d^n x \bar{C}_a (\partial^\mu A_\mu^a + \frac{1}{2} \alpha B^a). \quad (3.5)$$

The cohomology of the associated BRST differential $s = (S_{\text{gf}}, \cdot)_{\phi, \tilde{\phi}^*}$ in the space of local functions or in the space of local functionals is isomorphic to the cohomology of the canonical BRST differential in the respective spaces and can be obtained from it through the shift of antifields $\phi^* = \tilde{\phi}^* + \delta^L \Psi / \delta \phi$. The dependence of the gauge fixed action on the fields and antifields of the non minimal sector is explicitly given by

$$\begin{aligned} \frac{\delta^R S_{\text{gf}}}{\delta C^a} &= -(S_{\text{gf}}, \frac{\delta^R \Psi}{\delta C^a})_{\phi, \tilde{\phi}^*}, & \frac{\delta^R S_{\text{gf}}}{\delta B^a} &= -(S_{\text{gf}}, \frac{\delta^R \Psi}{\delta B^a})_{\phi, \tilde{\phi}^*} + \tilde{C}_a^*, \\ \frac{\delta^R S_{\text{gf}}}{\delta \tilde{C}_a^*} &= B^a, & \frac{\delta^R S_{\text{gf}}}{\delta B_a^*} &= 0. \end{aligned}$$

The dependence of the gauge fixed solution of the master equation S_{gf} on any parameter α^i appearing in the gauge fixing fermion Ψ alone, and not in the minimal solution S of the master equation, is given by

$$\frac{\partial S_{\text{gf}}}{\partial^R \alpha^i} = -(S_{\text{gf}}, \frac{\partial^R \Psi}{\partial \alpha^i}). \quad (3.6)$$

In the following, it will always be understood that the gauge fixed action with tilded antifields is used in manipulations involving the Green's functions, even if we do not always indicate this explicitly, when we are interested in statements concerning the local BRST cohomology.

4 Higher order cohomological restrictions and renormalization

4.1 Regularization

We will assume that there is a regularization with the properties of dimensional regularization as explained in reference [65], i.e.,

- the regularized action $S_\tau = \sum_{n=0} \tau^n S_n$ is a polynomial or a power series in τ , the τ independent part corresponding to the starting point action $S_0 = S$, so that the algebraic relations that hold for the classical action S hold in the regularized theory for S_0 .
- if the renormalization has been carried out up to $n - 1$ loops, the divergences of the effective action at n loops are poles in τ up to the order n with residues that are local functionals, and
- the regularized quantum action principle holds [56].

Let $\tilde{S} = S_\tau + \rho^* \theta_\tau$, with $\theta_\tau = \frac{1}{2\tau}(S_\tau, S_\tau)$, so that $\theta_0 = (S, S_1)$, and ρ^* a global source in ghost number -1 . On the classical level, we have, using $(\rho^*)^2 = 0$,

$$\frac{1}{2}(\tilde{S}, \tilde{S}) = \tau \frac{\partial \tilde{S}}{\partial \rho^*}, \quad (4.1)$$

$$(\tilde{S}, \frac{\partial \tilde{S}}{\partial \rho^*}) = 0, \quad (4.2)$$

which translates, according to the quantum action principle, into the corresponding equations for the regularized generating functional for 1PI vertex functions:

$$\frac{1}{2}(\tilde{\Gamma}, \tilde{\Gamma}) = \tau \frac{\partial \tilde{\Gamma}}{\partial \rho^*}, \quad (4.3)$$

$$(\tilde{\Gamma}, \frac{\partial \tilde{\Gamma}}{\partial \rho^*}) = 0. \quad (4.4)$$

Using $(\rho^*)^2 = 0$, these equations reduce to

$$\frac{1}{2}(\Gamma, \Gamma) = \tau \frac{\partial \tilde{\Gamma}}{\partial \rho^*}, \quad (4.5)$$

$$(\Gamma, \frac{\partial \tilde{\Gamma}}{\partial \rho^*}) = 0. \quad (4.6)$$

At one loop, we get

$$(S_\tau, \Gamma^{(1)}) = \tau \theta^{(1)}, \quad (4.7)$$

$$(S_\tau, \theta^{(1)}) + (\Gamma^{(1)}, \theta_\tau) = 0, \quad (4.8)$$

where $\Gamma^{(1)}$ and $\theta^{(1)}$ are respectively the one loop contributions of Γ and $\partial\tilde{\Gamma}/\partial\rho^*$. By assumption, we have

$$\Gamma^{(1)} = \sum_{n=-1} \tau^n \Gamma^{(1)n}, \quad (4.9)$$

$$\theta^{(1)} = \sum_{n=-1} \tau^n \theta^{(1)n}, \quad (4.10)$$

where $\Gamma^{(1)-1}$ and $\theta^{(1)-1}$ are local functionals.

4.2 Lowest order cohomological restrictions

At $1/\tau$, equations (4.7) and (4.8) give

$$(S, \Gamma^{(1)-1}) = 0, \quad (4.11)$$

$$(S, \theta^{(1)-1}) + (\Gamma^{(1)-1}, \theta_0) = 0. \quad (4.12)$$

Using $\theta_0 = (S, S_1)$ and equation (4.11), equation (4.12) reduces to

$$(S, \theta^{(1)-1} - (S_1, \Gamma^{(1)-1})) = 0. \quad (4.13)$$

In addition, we get, from the term independent of τ in equation (4.7),

$$(S, \Gamma^{(1)0}) = \theta^{(1)-1} - (S_1, \Gamma^{(1)-1}). \quad (4.14)$$

The term linear in τ gives

$$(S, \Gamma^{(1)1}) = \theta^{(1)0} - (S_1, \Gamma^{(1)0}) - (S_2, \Gamma^{(1)-1}). \quad (4.15)$$

The one loop renormalized effective action is $\Gamma_R^1 = S + \hbar\Gamma^{(1)0} + O(\hbar^2, \tau)$, where the notation $O(\hbar^2, \tau)$ means that those terms which are not of order at least two in \hbar are of order at least one in τ and vanish when the regularization is removed ($\tau \rightarrow 0$), so that

$$\frac{1}{2}(\Gamma_R^1, \Gamma_R^1) = \hbar A_1 + O(\hbar^2, \tau), \quad (4.16)$$

with, using equation (4.14),

$$A_1 = \theta^{(1)-1} - (S_1, \Gamma^{(1)-1}), \quad (4.17)$$

and the consistency conditions for local functionals

$$(S, \Gamma^{(1)-1}) = 0 \implies \Gamma^{(1)-1} = c_1^i C_i + (S, \Xi_1), \quad (4.18)$$

$$(S, A_1) = 0 \implies A_1 = a_1^i A_i + (S, \Sigma_1), \quad (4.19)$$

where $[C_i]$ and $[A_i]$ are respectively a basis of representatives for $H^0(s)$ and $H^1(s)$. These are the standard cohomological restrictions on the divergences and the anomalies of the quantum theory.

4.3 First order cohomological restrictions

The idea is now to get additional cohomological restrictions on anomalies and counterterms, using the bracket induced in cohomology discussed in section 2.1. In order to do so, we couple to the starting point action an arbitrary local BRST cohomological class.

Let D be a BRST cocycle in any ghost number g and consider $S^j = S + jD$, where the source j is of ghost number $-g$. The regularized action is $S_\tau^j = S_\tau + jD_\tau$, with D a polynomial in τ starting with D . If $\tilde{S}^j = S_\tau^j + \rho^* \theta_\tau^j$, where $\theta_\tau^j = \frac{1}{2\tau}(S_\tau, S_\tau) + \frac{1}{\tau}(jD_\tau, S_\tau)$, we have

$$\frac{1}{2}(\tilde{S}^j, \tilde{S}^j) = \tau \frac{\partial \tilde{S}^j}{\partial \rho^*} + O(j^2), \quad (4.20)$$

and the corresponding equation for the regularized generating functional $\tilde{\Gamma}_j$

$$\frac{1}{2}(\tilde{\Gamma}_j, \tilde{\Gamma}_j) = \tau \frac{\partial \tilde{\Gamma}_j}{\partial \rho^*} + O(j^2). \quad (4.21)$$

At one loop, we get for the term independent of ρ^* ,

$$(\Gamma_j^{(1)}, S_\tau^j) = \tau \theta_j^{(1)} + O(j^2). \quad (4.22)$$

The term linear in j of order $\frac{1}{\tau}$ gives

$$(D^{(1)-1}, S) + (D, \Gamma^{(1)-1}) = 0, \quad (4.23)$$

with $D^{(1)-1} = (\partial \Gamma_j^{(1)-1} / \partial j)|_{j=0}$. This gives our first theorem.

Theorem 1 *The antibracket of the divergent one loop part $\Gamma^{(1)-1}$, which is BRST closed and local, with any local BRST cocycle is BRST exact in the space of local functionals.*

The theorem can be reformulated by saying that the antibracket map induced in the local BRST cohomology groups

$$([\Gamma^{(1)-1}], [D])_M = [0] \quad (4.24)$$

for all $[D] \in H^g(s)$. This equation represents a cohomological restriction on the coefficients c_1^i that can appear ; it can be calculated classically from the knowledge of $H^0(s)$ and the antibracket map from $H^0(s) \otimes H^g(s)$ to $H^{g+1}(s)$. According to the previous section, the theorem holds in particular when $D = \Gamma^{(1)-1}$ or $D = A_1$.

In the same way, the consistency condition is

$$(\Gamma_j, \frac{\partial \tilde{\Gamma}_j}{\partial \rho^*}) + O(j^2) = 0, \quad (4.25)$$

and gives at one loop,

$$(\Gamma_j^{(1)}, \theta_\tau^j) + (S_\tau^j, \theta_j^{(1)}) + O(j^2) = 0. \quad (4.26)$$

The term linear in j of order $\frac{1}{\tau}$ gives

$$\begin{aligned} & (D^{(1)-1}, \theta_0) - \left(\frac{\partial \theta_0^j}{\partial j} \Big|_{j=0}, \Gamma^{(1)-1} \right) \\ & + (D, \theta^{(1)-1}) - \left(\frac{\partial \theta_j^{(1)-1}}{\partial j} \Big|_{j=0}, S \right) = 0. \end{aligned} \quad (4.27)$$

Using $\theta_0 = (S, S_1)$, $\partial \theta_0^j / \partial j|_{j=0} = (D_1, S) + (D, S_1)$, equations (4.11), (4.17) and (4.23), we get

$$(D, A_1) - \left(\frac{\partial \theta_j^{(1)-1}}{\partial j} \Big|_{j=0} - (D_1, \Gamma^{(1)-1}) - (D^{(1)-1}, S_1), S \right) = 0. \quad (4.28)$$

This gives our second result.

Theorem 2 *The antibracket of the BRST closed first order anomaly A_1 with any local BRST cocycle is BRST exact in the space of local functionals.*

The theorem can again be reformulated by saying that the antibracket map

$$([A_1], [D])_M = [0] \quad (4.29)$$

for all $[D] \in H^g(s)$; it represents a classical cohomological restriction on the coefficients a_1^i that can appear.

4.4 Higher orders

Let $B^0 = S$ and $B^1 = \Gamma^{(1)-1}$. We have the following theorem.

Theorem 3 *The first order counterterms can be completed into a local deformation of S , i.e., there exist local functionals B^n such that*

$$\frac{1}{2}(S^{j^\infty}, S^{j^\infty}) = 0, \quad (4.30)$$

$$S^{j^\infty} = S + \sum_{n=1} j^n B^n. \quad (4.31)$$

Proof. The theorem is true for j^0, j^1 and j^2 , if we take $D = \Gamma^{(1)-1} = B^1$ in (4.23) and $B^2 = 1/2(\partial \Gamma_j^{(1)-1} / \partial j)|_{j=0}$. Suppose the theorem true at order j^k i.e., we have

$$\frac{1}{2}(S^{j^k}, S^{j^k}) = O(j^{k+1}), \quad (4.32)$$

$$S^{j^k} = S + \sum_{n=1}^k j^n B^n. \quad (4.33)$$

and

$$B^n = \frac{1}{n}(\partial^{n-1}\Gamma_{j^{n-1}}^{(1)-1}/\partial j^{n-1})|_{j=0}. \quad (4.34)$$

At the regularized level, consider the action

$$S_\tau^{j^k} = S_\tau + \sum_{n=1}^k j^n B_\tau^n \quad (4.35)$$

and $\tilde{S}^{j^k} = S_\tau^{j^k} + \rho^* \theta_\tau^{j^k}$, with $\theta_\tau^{j^k} = \frac{1}{2\tau}(S_\tau^{j^k}, S_\tau^{j^k}) + O(j^{k+1})$, so that

$$\frac{1}{2}(\tilde{S}^{j^k}, \tilde{S}^{j^k}) = \tau \frac{\partial \tilde{S}^{j^k}}{\partial \rho^*} + O(j^{k+1}). \quad (4.36)$$

The corresponding equation for $\tilde{\Gamma}_{j^k}$ based on the action \tilde{S}^{j^k} is

$$\frac{1}{2}(\tilde{\Gamma}_{j^k}, \tilde{\Gamma}_{j^k}) = \tau \frac{\partial \tilde{\Gamma}_{j^k}}{\partial \rho^*} + O(j^{k+1}). \quad (4.37)$$

At one loop, we get, for the part independent of ρ^* ,

$$(S_\tau^{j^k}, \Gamma_{j^k}^{(1)}) = \tau \theta_{j^k}^{(1)} + O(j^{k+1}). \quad (4.38)$$

At order j^k , this equation gives

$$\begin{aligned} & \left(S_\tau, \frac{\partial^k \Gamma_{j^k}^{(1)}}{\partial j^k} \Big|_{j=0} \right) + \left(B_\tau^1, \frac{\partial^{k-1} \Gamma_{j^k}^{(1)}}{\partial j^{k-1}} \Big|_{j=0} \right) + \dots \\ & + \left(B_\tau^k, \Gamma_{j^k}^{(1)} \Big|_{j=0} \right) = \tau \frac{\partial^k \theta_{j^k}^{(1)}}{\partial j^k} \Big|_{j=0}. \end{aligned} \quad (4.39)$$

At order $1/\tau$, we get, using

$$\frac{\partial^{n-1} \Gamma_{j^k}^{(1)-1}}{\partial j^{n-1}} \Big|_{j=0} = \frac{\partial^{n-1} \Gamma_{j^{n-1}}^{(1)-1}}{\partial j^{n-1}} \Big|_{j=0} = n B^n, \quad (4.40)$$

for $n = 1, \dots, k-1$ and defining $\frac{\partial^k \Gamma_{j^k}^{(1)-1}}{\partial j^k} \Big|_{j=0} = (k+1) B^{k+1}$, the relation

$$(S, (k+1) B^{k+1}) + (B^1, k B^k) + \dots + (B^k, B^1) = 0, \quad (4.41)$$

or equivalently

$$0 = \sum_{m=0}^k (B^m, (k+1-m) B^{k+1-m}) = \frac{(k+1)}{2} \sum_{m=0}^{k+1} (B^m, B^{k+1-m}), \quad (4.42)$$

which proves the theorem. \square

Let $E^0 = A_1 = \theta^{(1)-1} - (B^1, S_1)$.

Theorem 4 *The lowest order contribution to the anomaly E^0 can be extended to a local cocycle of the deformed solution of the master equation S^{j^∞} , i.e., there exist local functionals E^m such that*

$$(S^{j^\infty}, E^{j^\infty}) = 0, \quad (4.43)$$

$$E^{j^\infty} = \sum_{m=0} j^m E^m. \quad (4.44)$$

Proof. The theorem holds for j^0 and j^1 by taking in (4.28) $D = B^1$, and defining

$$E^1 = \left. \frac{\partial \theta_j^{(1)-1}}{\partial j} \right|_{j=0} - (D_1, \Gamma^{(1)-1}) - (D^{(1)-1}, S_1) \quad (4.45)$$

$$= \left. \frac{\partial \theta_j^{(1)-1}}{\partial j} \right|_{j=0} - (B_1, B_1^1) - (2B^2, S_1). \quad (4.46)$$

Let us define

$$E^m = \left. \frac{\partial^m \theta_{j^m}^{(1)-1}}{\partial j^m} \right|_{j=0} - \sum_{n=0}^m ((n+1)B^{n+1}, B_1^{m-n}). \quad (4.47)$$

The consistency condition is

$$(\Gamma_{j^k}, \frac{\partial \tilde{\Gamma}_{j^k}}{\partial \rho^*}) = O(j^{k+1}). \quad (4.48)$$

At one loop, we have,

$$(\Gamma_{j^k}^{(1)}, \theta_\tau^{j^k}) + (S_\tau^{j^k}, \theta_{j^k}^{(1)}) = O(j^{k+1}). \quad (4.49)$$

The term of order j^k of this equation gives

$$\sum_{m=0}^k \left[\left(\left. \frac{\partial^m \Gamma_{j^k}^{(1)}}{\partial j^m} \right|_{j=0}, \left. \frac{\partial^{k-m} \theta_\tau^{j^k}}{\partial j^{k-m}} \right|_{j=0} \right) + \left(B_\tau^m, \left. \frac{\partial^{k-m} \theta_{j^k}^{(1)}}{\partial j^{k-m}} \right|_{j=0} \right) \right] = 0. \quad (4.50)$$

At order $1/\tau$, we get

$$\begin{aligned} \sum_{m=0}^k \left[\left((m+1)B^{m+1}, \sum_{l=0}^{k-m} (B^l, B_1^{k-m-l}) \right) \right. \\ \left. + \left(B^m, \left. \frac{\partial^{k-m} \theta_{j^k}^{(1)-1}}{\partial j^{k-m}} \right|_{j=0} \right) \right] = 0. \end{aligned} \quad (4.51)$$

Using the Jacobi identity, the first term is given by

$$\sum_{m=0}^k \sum_{l=0}^{k-m} [(((m+1)B^{m+1}, B^{k-m-l}), B_1^l) - (B^l, ((m+1)B^{m+1}, B_1^{k-m-l}))]. \quad (4.52)$$

Changing the sum $\sum_{m=0}^k \sum_{l=0}^{k-m}$ to the equivalent sum $\sum_{l=0}^k \sum_{m=0}^{k-l}$, the first term of this equation vanishes on account of (4.42), while the second term, using the definition (4.47), combines with the second term of (4.51) to give

$$\sum_{m=0}^k (B^m, E^{k-m}) = 0, \quad (4.53)$$

which proves the theorem. \square

The investigation in this section is a first step in order to analyze the cohomological restrictions on anomalies and counterterms at higher orders in \hbar . To see this, we note that if we put $j = (-\hbar/\tau)$, the action $S^{(-\hbar/\tau)\infty}$ satisfies the (deformed) master equation $1/2(S^{(-\hbar/\tau)\infty}, S^{(-\hbar/\tau)\infty}) = 0$, while the corresponding effective action is finite at order \hbar . Its divergences at order \hbar^2 are poles up to order 2 in τ with residues that are local functionals. A systematic analysis of the subtraction procedure at higher orders in \hbar will be presented in the context of the extended antifield formalism below.

4.5 Cohomological restrictions through mixed antibracket map

In the antifield formalism, there is an additional map relating two different types of cohomologies, the BRST cohomology for local functionals $H^{*,n}(s|d, \Omega)$ and the BRST cohomology for local functions, $H^*(s, \Omega^0)$. It is defined in terms of the bracket $(\cdot, \cdot)_{alt}$ from local functionals tensor product with local functions to local functions defined by

$$(\cdot, \cdot)_{alt} : H^n(d, \Omega) \otimes \Omega^0 \longrightarrow \Omega^0, \quad (4.54)$$

$$(A, b)_{alt} = \sum_{k=0} \partial_{\mu_1} \dots \partial_{\mu_k} \frac{\delta^R A}{\delta \phi^a} \frac{\partial^L b}{\partial \phi_{a, (\mu_1 \dots \mu_k)}^*} - \sum_{k=0} \partial_{\mu_1} \dots \partial_{\mu_k} \frac{\delta^R A}{\delta \phi_a^*} \frac{\partial^L b}{\partial \phi_{, (\mu_1 \dots \mu_k)}^a}, \quad (4.55)$$

where $A = \int d^n x a$. It is straightforward to verify that this bracket induces a well defined mixed map $(\cdot, \cdot)_m$ in cohomology, i.e., that it maps cocycles to cocycles and that the resulting cohomology class does not depend on the choice of the representatives:

$$(\cdot, \cdot)_m : H^{g_1, n}(s|d, \Omega) \otimes H^{g_2}(s, \Omega^0) \longrightarrow H^{g_1+g_2+1}(s, \Omega^0), \quad (4.56)$$

$$([A], [b])_m = [(A, b)_{alt}]. \quad (4.57)$$

By using a local source $j(x)$ instead of a coupling constant j to couple the representative d of a class $[d] \in H^g(s, \Omega^0)$, theorems 1 and 2 of section 4.3 become

Theorem 5 *The mixed antibracket map of the first order divergences or of the first order anomalies with any non integrated BRST cohomology class $[d] \in H^g(s, \Omega^0)$ vanishes:*

$$([\Gamma^{(1)-1}], [d])_m = 0, \quad (4.58)$$

$$([A_1], [d])_m = 0. \quad (4.59)$$

4.6 Application 1: Elimination of antifield dependent counterterms in Yang-Mills theories with $U(1)$ factors

In this section, we will discuss the elimination by higher order cohomological restrictions, imposed by the mixed antibracket map of the previous subsection, of a type of antifield dependent counterterms arising in non semi-simple Yang-Mills theories. These counterterms have been discussed for the first time in [19, 20], were analyzed from a cohomological point of view in [17, 18] and reconsidered in the concrete context of the standard model in [85]. These counterterms (also called instabilities because they are not present in the starting point action) have the following general structure [17, 18]:

$$K' = f_\alpha^\Delta \int d^n x j_\Delta^\mu A_\mu^\alpha + (A_a^{*\mu} X_{\mu\Delta}^a + y_i^* X_\Delta^i) C^\alpha,$$

where f_α^Δ are constants, A_μ^α abelian gauge fields, j_Δ^μ non trivial conserved currents and $\delta_\Delta A_\mu^a = X_{\mu\Delta}^a$, $\delta_\Delta y^i = X_\Delta^i$ the generators of the corresponding symmetries on all the gauge fields A_μ^a and the matter fields y^i . In order to eliminate these instabilities by cohomological means, we will show that:

It is sufficient that there exists a set of local, non integrated, off-shell gauge invariant polynomials $O_\Gamma(x)$ constructed out of the A_μ^a, y^i and their derivatives, that break the global symmetries δ_Δ in the following sense: the variation of $O_\Gamma(x)$ under the gauged global symmetries δ_Δ with gauge parameter given by $f_\alpha^\Delta \epsilon^\alpha$ should not be equal on shell to an ordinary gauge transformation (involving the abelian gauge parameters ϵ^α alone) of some local polynomials $P_\Gamma(x)$ constructed out of the A_μ^a, y^i and their derivatives.

Indeed, using the extended action ⁵ $S_{k(x)} = S + \int d^n x k^\Gamma(x) O_\Gamma(x)$, which satisfies $1/2(S_{k(x)}, S_{k(x)}) = 0$ and the corresponding regularized action principle, it follows from the equation independent of the sources $k(x)$ that the divergences $\Gamma^{(1)}_{div}$ of the theory without $k(x)$ are, as usual, required to be BRST invariant. The terms linear in $k(x)$ then imply

$$\left(\frac{\delta \Gamma^{(1)}_{k(x)div}}{\delta k^\Gamma(x)} \right) \Big|_{k(x)=0}, S) + (O_\Gamma(x), \Gamma^{(1)}_{div}) = 0, \quad (4.60)$$

or, equivalently,

$$([\Gamma^{(1)}_{div}], [O_\Gamma])_m = 0. \quad (4.61)$$

The second term of this equation gives for the antifield dependent counterterms $\Gamma^{(1)}_{div} = K'$ above $(O_\Gamma(x), K') = (C^\alpha f_\alpha^\Delta \delta_\Delta) O_\Gamma(x)$, because we have $O_\Gamma(x)$ can be chosen to be independent of the antifields. From (4.60), it then follows that $(C^\alpha f_\alpha^\Delta \delta_\Delta) O_\Gamma(x)$ must be given on-shell by a gauge transformation, involving the abelian ghosts alone, of polynomials $P_\Gamma(x)$. This follows by using the explicit form

⁵The author thanks P.A. Grassi for suggesting the use of external sources instead of external couplings in this example.

of the BRST differential, and after evaluation, putting to zero the antifields, and the non abelian ghosts. Hence, the counterterms K' are excluded a priori whenever it is possible to construct $O_\Gamma(x)$'s for which the corresponding $P_\Gamma(x)$ do not exist so that (4.60) cannot be satisfied.

Remark: In this example, we use external sources and the mixed antibracket map instead of coupling constants and the standard antibracket map because the restrictions we get are stronger and the discussion is simplified: we need not worry about possible integrations by parts (in momentum space, this means that the restrictions we get are valid for all values of the external momentum and not only for zero external momentum).

The condition (4.61) means that besides the arguments of [19, 20, 42, 85], there exists an elegant cohomological mechanism to eliminate this type of antifield dependent counterterms.

In the concrete case of the standard model, the global symmetries δ_Δ correspond to lepton and baryon number conservation. There is only one abelian ghost C^α , the abelian gauge transformation of the matter fields being $\delta_{abelian}y = i\mathcal{Y}y$, where $\mathcal{Y} = \mathcal{Y}_j^i y^j \frac{\partial}{\partial y^i}$ is the hypercharge. As an example of O_Γ 's we can take any three linearly independent operators out of the lepton number non conserving gauge invariant operators of dimension 5 in the matter fields given in eq.(20) of [86] (they can also be found in eq. (21.3.54) of [87]) and one baryon number non conserving operator out of the six dimension 6 gauge invariant operators given in eqs. (1)-(6) in [86, 88]. Because these operators are build out of the undifferentiated matter fields alone, a sufficient condition for (4.60) to hold is the existence of $P'_\Gamma(x)$'s build out of the undifferentiated y^i such that

$$f^\Gamma n_\Gamma O_\Gamma = \mathcal{Y}P'_\Gamma, \quad (4.62)$$

(with no summation over Γ), where n_Γ is the lepton number of the O_Γ 's for $\Gamma = 1, 2, 3$ and the baryon number for O_4 . This follows by identifying the term in the abelian ghost and putting, in addition to the non abelian ghosts and the antifields, the derivatives of the abelian ghost, the derivatives of the matter fields and all the gauge fields to zero and using the fact that the equations of motion necessarily involve derivatives. Because the O_Γ 's we have chosen are all of homogeneity 4 in the y^i and \mathcal{Y} is of homogeneity 0, we can assume that the homogeneity of the P'_Γ 's is also 4. By decomposing the space M_4 of monomials of homogeneity 4 in the y^i into eigenspaces of the hermitian operator \mathcal{Y} with definite eigenvalues $M_4 = M_4^0 + \oplus_{n \neq 0} M_4^n$, it follows that (4.62) has no non trivial solutions. Indeed, decomposing $P'_\Gamma = P_\Gamma^{0'} + \sum_{n \neq 0} P_\Gamma^{n'}$, (4.62) reads $f^\Gamma n_\Gamma O_\Gamma = \sum_{n \neq 0} n P_\Gamma^{n'}$. Applying \mathcal{Y} k times and using the fact that gauge invariance of O_Γ implies $\mathcal{Y}O_\Gamma = 0$, we get $\sum_{n \neq 0} n^k P_\Gamma^{n'} = 0$. We then can conclude that $P_\Gamma^{n'} = 0$ for $n \neq 0$, which implies $f^\Gamma = 0$.

As usual, this one loop reasoning can be extended recursively to higher orders, or alternatively, it can be discussed independently of the assumption that there exists an invariant regularization scheme in the context of algebraic renormalization.

5 Extended antifield formalism. Classical theory

5.1 Coupling constants

The solution S of the classical master equation usually depends on some coupling constants. Differentiating (3.4) with respect to such a coupling constant g , implies that $(S, \frac{\partial^R S}{\partial g}) = 0$, so that $[\frac{\partial^R S}{\partial g}] \in H^0(s)$. Note that the presence of coupling constants implies that the cohomology of s has to be computed in the space of local functionals depending on the coupling constants. We will not specify more precisely the functional dependence on these couplings, although in the applications below, we have in mind mostly a polynomial or a formal power series dependence.

Let us adapt the considerations in [89] (see also [90] chapter 7.7) to the present context.

Definition 1 *A set of coupling constants g^i is essential iff the relation $\frac{\partial^R S}{\partial g^i} \lambda^i = (S, \Xi)$ implies $\lambda^i = 0$, where λ^i may depend on all the couplings of the theory.*

In other words, essential couplings correspond to independent elements $[\frac{\partial^R S}{\partial g^i}]$ of $H^0(s)$ computed in the space of local functionals over the ring of functions in the couplings. (In this context, one does not want to consider as independent cohomology classes local functionals that differ only by a factor depending on the couplings alone.) It follows that essential couplings stay essential after anticanonical field-antifield redefinitions, because these redefinitions do not affect the cohomology.

In the following, we suppose that S depends only on essential couplings. Note that because of equation (3.6), the couplings introduced through the gauge fixing alone are all redundant.

5.2 Application of the main theorem

Let us now apply and recall the results of sections 2.2 and 2.3 in the present case.

5.2.1 Anti constant ghosts and acyclic differentials

Let $\{[S_A]\}$ be a basis of $H^*(s, \mathcal{F})$ over the ring of functions in the essential coupling constants of the theory, so that the equation $(S, A) = 0$ implies $A = S_A \lambda^A + (S, B)$, where λ^A is independent of the fields and anti-fields, but can depend on the coupling constants of the theory, with $S_A \lambda^A + (S, B) = 0$ iff $\lambda^A = 0$. For each S_A of the above basis, we introduce a constant “ghost” ξ^A and a constant “antifield” ξ_A^* such that $gh \xi^A = -gh S_A$, $gh \xi_A^* = -gh \xi^A - 1$. We consider the space \mathcal{E} of functionals \mathcal{A} of the form

$$\mathcal{A} = A[\phi, \phi^*, \xi] + \xi_A^* \lambda^A(\xi), \quad (5.1)$$

i.e., \mathcal{A} contains a local functional A which admits in addition to the dependence on the coupling constants, a dependence on the constant ghosts ξ^A , and a non integrated piece linear in the constant antifields ξ_A^* depending only on the constant ghosts (and the coupling constants).

The differential $\tilde{\delta}$ is defined by $\tilde{\delta}A = (S, A)$, $\tilde{\delta}\xi_A^* = S_A$, and $\tilde{\delta}\xi^A = 0$.

Corollary 1 *The cohomology of $\tilde{\delta}$ is trivial, $H(\tilde{\delta}, \mathcal{E}) = 0$.*

We define the resolution degree to be the degree in the ghosts ξ^A , which implies that $\tilde{\delta}$ is of degree 0.

The extended antibracket is defined by

$$\begin{aligned} (\cdot, \cdot) &= (\cdot, \cdot) + (\cdot, \cdot)_\xi \\ &= (\cdot, \cdot) + \frac{\partial^R}{\partial \xi^A} \frac{\partial^L}{\partial \xi_A^*} - \frac{\partial^R}{\partial \xi_A^*} \frac{\partial^L}{\partial \xi^A} \end{aligned} \quad (5.2)$$

and satisfies the same graded antisymmetry and graded Jacobi identity as the usual antibracket. The extended antibracket has two pieces, the old piece (\cdot, \cdot) , which is of degree 0, and the new piece $(\cdot, \cdot)_\xi$, which is of degree -1 .

Corollary 2 *There exists a solution $\tilde{\mathcal{S}} \in \mathcal{E}$ of ghost number 0 to the master equation*

$$\frac{1}{2}(\tilde{\mathcal{S}}, \tilde{\mathcal{S}}) = 0. \quad (5.3)$$

with initial condition $\tilde{\mathcal{S}} = S + S_A \xi^A + \dots$, where the dots denote terms of resolution degree higher or equal to 2. The cohomology of the differential $\tilde{s} = (\tilde{\mathcal{S}}, \cdot)$ in \mathcal{E} is trivial.

The solution $\tilde{\mathcal{S}}$ is of the form

$$\tilde{\mathcal{S}} = S + \sum_{k=1} S_{A_1 \dots A_k} \xi^{A_1} \dots \xi^{A_k} + \sum_{m=2} \xi_B^* f_{A_1 \dots A_m}^B \xi^{A_1} \dots \xi^{A_m}, \quad (5.4)$$

which implies the graded symmetry of the generalized structure constants $f_{A_1 \dots A_m}^B$ and the functionals $S_{A_1 \dots A_k}$. The ξ_A^* independent part of the master equation (5.3) gives, at resolution degree $r \geq 1$, the relations

$$\begin{aligned} (S, S_{A_1 \dots A_r}) + \sum_{k=1}^{r-1} \frac{1}{2} (S_{(A_1 \dots A_k}, S_{A_{k+1} \dots A_r})) (-)^{(A_1 + \dots + A_k)(A_{k+1} \dots A_r + 1)} \\ + \sum_{k=1}^{r-1} k S_{(A_1 \dots A_{k-1} | B |} f_{A_k \dots A_r}^B = 0, \end{aligned} \quad (5.5)$$

where $(\)$ denotes graded symmetrization. The first relations read explicitly

$$(S, S_{A_1}) = 0, \quad (5.6)$$

$$(S, S_{A_1 A_2}) + \frac{1}{2} (S_{A_1}, S_{A_2}) (-)^{A_1(A_2+1)} + S_B f_{A_1 A_2}^B = 0, \quad (5.7)$$

$$\begin{aligned} (S, S_{A_1 A_2 A_3}) + (S_{(A_1}, S_{A_2 A_3})) (-)^{A_1(A_2+A_3+1)} \\ + S_B f_{A_1 A_2 A_3}^B + 2 S_{(A_1 | B |} f_{A_2 A_3}^B = 0, \end{aligned} \quad (5.8)$$

\vdots

The ξ_A^* dependent part of the master equation (5.3) gives, for $r \geq 3$, the generalized Jacobi identities

$$\sum_{m=2}^{r-1} m f_{(A_1 \dots A_{m-1} | B|}^C f_{A_m \dots A_r}^B = 0, \quad (5.9)$$

the first identities being

$$2f_{(A_1|B|}^C f_{A_2 A_3}^B = 0, \quad (5.10)$$

$$2f_{(A_1|B|}^C f_{A_2 A_3 A_4}^B + 3f_{(A_1 A_2|B|}^C f_{A_3 A_4}^B = 0, \quad (5.11)$$

\vdots

5.2.2 Ambiguity of the construction

The above solution $\tilde{\mathcal{S}}$ is not unique. For a given initial condition, there is at each stage of the construction of $\tilde{\mathcal{S}}$, for $k \geq 2$, the liberty to add the exact term $\tilde{\delta} K_k$ to $\tilde{\mathcal{S}}_k$. While this liberty will not affect the structure constants of order k , since a $\tilde{\delta}$ exact term does not involve a ξ^* dependent term, it will in general affect the structure constants of order strictly higher than k . Furthermore, there is a freedom in the choice of the initial condition: instead of $S_1 = S_A \xi^A$, one could have chosen $S'_1 = \sigma_A^B S_B \xi^A + (S, K_A) \xi^A$ with an invertible matrix σ_A^B . If we consider the following anticanonical redefinitions:

$$z' = \exp(\cdot, K_A \xi^A) z, \quad (5.12)$$

$$\xi'^B = \sigma_A^B \xi^A, \quad \xi'^*_B = \sigma^{-1A}_B \xi^*_A, \quad (5.13)$$

we have that $S + S'_1 = S(z') + S_B(z') \xi'^B + O(\xi^2)$. We can then consider the solution $\tilde{\mathcal{S}}'$ in terms of the new variables. This is equivalent to taking as initial condition $S(z') + S_B(z') \xi'^B$ and making the same choices for the terms of degree higher than 2 in the new variables than we did before in the old variables. It is thus always possible to make the choices in the construction of $\tilde{\mathcal{S}}$ for $k \geq 2$ in such a way that the structure constants $f_{A_1 \dots A_m}^B$ do not depend on the choice of representatives for the cohomology classes and transform tensorially with respect to a change of basis in $H^*(s, \mathcal{F})$. Hence, we have shown

Corollary 3 *Associated to a solution $\tilde{\mathcal{S}}$ of the master equation (5.3), there exist multi-linear, graded symmetric maps in cohomology, defined through the structure constants $f_{A_1 \dots A_r}^B$:*

$$l_r : \wedge^r H^*(s) \longrightarrow H^*(s) \quad (5.14)$$

$$l_r([S_{A_1}], \dots, [S_{A_r}]) = [S_B] f_{A_1 \dots A_r}^B \quad (5.15)$$

5.2.3 Essential couplings and constant ghosts

In the construction so far, there has been a kind of redundancy because we have coupled to the solution of the master equation with new independent couplings all

local BRST cohomological classes, although the classes $[\frac{\partial^R S}{\partial g^i}]$ in ghost number 0 are already coupled through the essentials couplings g^i .

Since the $[\frac{\partial^R S}{\partial g^i}]$ are linearly independent, one can construct a basis $[S_A]$ of $H^*(s)$ such that the $[\frac{\partial^R S}{\partial g^i}]$ are the first elements. Let us denote the remaining elements by $[S_\alpha]$, so that $\{[S_A]\} = \{[\frac{\partial^R S}{\partial g^i}], [S_\alpha]\}$. The construction of the generating functional $\tilde{\mathcal{S}}$ then starts with $S(g^i) + \frac{\partial^R S}{\partial g^i} \xi^i + S_\alpha \xi^\alpha$.

Consider the action $\bar{S} = S(g^i + \xi^i)$. A basis of the cohomology of \bar{S} is given by $\{[\frac{\partial^R \bar{S}}{\partial g^i}], [\bar{S}_\alpha]\}$, with associated differential $\tilde{\delta} = (\bar{S}, \cdot) + \frac{\partial^R \bar{S}}{\partial \xi^i} \frac{\partial^L}{\partial \xi_i^*} + \bar{S}_\alpha \frac{\partial^L}{\partial \xi_\alpha^*}$, which is acyclic in the space where the only dependence on ξ^i is through the combination $g^i + \xi^i$. If we take as starting point the action $\bar{S} + \bar{S}_\alpha \xi^\alpha$ and start the perturbative construction of the solution of the master equation, with resolution degree the degree in the ghost ξ^α alone, the ghosts ξ^i only appear through the combination $g^i + \xi^i$, because of the properties of $\tilde{\delta}$. The solution $\tilde{\mathcal{S}}$ will then be of the form

$$\tilde{\mathcal{S}} = \bar{S} + \sum_{k=1} \bar{S}_{\alpha_1 \dots \alpha_k} \xi^{\alpha_1} \dots \xi^{\alpha_k} + \sum_{m=2} (\xi_\beta^* \bar{f}_{\alpha_1 \dots \alpha_m}^\beta + \xi_i^* \bar{f}_{\alpha_1 \dots \alpha_m}^i) \xi^{\alpha_1} \dots \xi^{\alpha_m}, \quad (5.16)$$

where the $\bar{S}_{\alpha_1 \dots \alpha_k}$, $\bar{f}_{\alpha_1 \dots \alpha_m}^i$, $\bar{f}_{\alpha_1 \dots \alpha_m}^i$ depend on the combination $g^i + \xi^i$.

Now, the solution $\tilde{\mathcal{S}}$ satisfies the initial condition $\tilde{\mathcal{S}}^1 = S(g) + \frac{\partial^R S}{\partial g^i} \xi^i + S_\alpha \xi^\alpha$ in the old resolution degree and the master equation (5.3). We can then derive the higher order maps l_r from the solution (5.16) and get

$$\begin{aligned} l_r([\frac{\partial^R S}{\partial g^{i_1}}], \dots, [\frac{\partial^R S}{\partial g^{i_n}}], [S_{\alpha_{n+1}}], \dots, [S_{\alpha_r}]) \\ = \frac{1}{n!} [S_\beta] \frac{\partial^{Rn} \bar{f}_{\alpha_{n+1} \dots \alpha_r}^\beta}{\partial g^{i_n} \dots \partial g^{i_1}}(g) + \frac{1}{n!} [\frac{\partial^R S}{\partial g^j}] \frac{\partial^{Rn} \bar{f}_{\alpha_{n+1} \dots \alpha_r}^j}{\partial g^{i_n} \dots \partial g^{i_1}}(g). \end{aligned} \quad (5.17)$$

In the following, we will make the redefinition $g^i + \xi^i \longrightarrow \xi^i$, and identify the essential couplings with some of the constant ghosts. Alternatively, the remaining constant ghosts can be considered as generalized essential coupling constants since they couple the remaining BRST cohomology classes, which play the role of generalized observables in this formalism.

5.2.4 Decomposition of \tilde{s}

The space \mathcal{E} admits the direct sum decomposition $\mathcal{E} = F \oplus G$, where $F = \mathcal{E}|_{\xi^*=0}$ is the space of functionals in the field and antifields with ξ dependence, but no ξ^* dependence, while G is the space of power series in ξ with a linear ξ^* dependence.

The differential \tilde{s} in \mathcal{E} induces two well-defined differentials, \bar{s} in F and s_Q in G given explicitly by

$$\bar{s} = (S(\xi), \cdot) + (-)^D f^D \frac{\partial^L}{\partial \xi^D} \quad (5.18)$$

and

$$s_{\Delta^*} = (\Delta^*, \cdot)_\xi, \quad \Delta^* = \xi_C^* f^C(\xi). \quad (5.19)$$

Indeed, for $\mathcal{A} = A(\xi) + \xi_D^* \lambda^D(\xi)$, the master equation (5.3) implies $(\tilde{\mathcal{S}}, (\tilde{\mathcal{S}}, \mathcal{A})) = 0$ and hence $(\tilde{\mathcal{S}}, \bar{s}A(\xi) + s_Q \xi_D^* \lambda^D(\xi)) = 0$ and then $(\bar{s})^2 A(\xi) + (s_{\Delta^*})^2 \xi_D^* \lambda^D(\xi) = 0$, which splits into two equations because the decomposition of \mathcal{E} is direct. If $\tilde{S} = S(\xi) + \xi_C^* f^C(\xi)$, $\mathcal{A} = A(\xi) + \xi_C^* \lambda^C(\xi)$ and $\mathcal{B} = B(\xi) + \xi_C^* \mu^C(\xi)$. The extended master equation (5.3) can be written compactly as

$$\frac{1}{2}(S(\xi), S(\xi)) + \frac{\partial^R S(\xi)}{\partial \xi^C} f^C = 0, \quad (5.20)$$

$$\frac{1}{2}(\xi_C^* f^C(\xi), \xi_D^* f^D(\xi))_\xi = 0, \quad (5.21)$$

so that (5.20) summarizes (5.5) and (5.21), which is equal to $\frac{1}{2}(\Delta^*, \Delta^*)_\xi = 0$, or explicitly $\frac{\partial^R f^D(\xi)}{\partial \xi^C} f^C(\xi) = 0$, summarizes the generalized Jacobi identities (5.9).

Corollary 4 *The cohomology groups $H(\bar{s}, F)$ and $H(s_{\Delta^*}, G)$ are isomorphic.*

More precisely,

$$\bar{s}A(\xi) = 0 \iff \begin{cases} A(\xi) = \bar{s}B(\xi) + \frac{\partial^R S(\xi)}{\partial \xi^C} \mu^C(\xi), \\ (\xi_C^* f^C(\xi), \xi_D^* \mu^D(\xi))_\xi = 0, \end{cases} \quad (5.22)$$

and

$$\begin{cases} \bar{s}B(\xi) + \frac{\partial^R S(\xi)}{\partial \xi^C} \mu^C(\xi) = 0, \\ (\Delta^*, \xi_B^* \mu^B(\xi))_\xi = 0, \end{cases} \iff \begin{cases} B(\xi) = \bar{s}C(\xi) + \frac{\partial^R S(\xi)}{\partial \xi^C} \nu^C(\xi), \\ \xi_D^* \mu^D(\xi) = (\Delta^*, \xi_E^* \nu^E(\xi))_\xi, \end{cases} \quad (5.23)$$

so that

$$\begin{aligned} m : H(s_{\Delta^*}, G) &\longrightarrow H(\bar{s}, F), \\ m([\xi_D^* \mu^D(\xi)]) &= [\frac{\partial^R S(\xi)}{\partial \xi^C} \mu^C(\xi)] \end{aligned} \quad (5.24)$$

is one-to-one and onto.

5.2.5 Discussion

(i) In order to compare the starting point cohomology $H^*(s, \mathcal{F})$ with the cohomology $H^*(\bar{s}, F)$, we can put the additional couplings ξ^α to zero in (5.22). The cocycle condition then reduces to the standard cocycle condition of the non extended formalism, $sA_{\xi^\alpha=0} = 0$. The same operation in the general solution gives $A_{\xi^\alpha=0} = sB_{\xi^\alpha=0} + \frac{\partial^R S}{\partial \xi^i} \mu_{\xi^\alpha=0}^i + S_\alpha \mu_{\xi^\alpha=0}^\alpha$. Contrary to the ordinary s cohomology, the coefficients $\mu_{\xi^\alpha=0}^A$ are not free however, but they come from μ^A 's which are constrained to satisfy the cocycle condition in (5.22). In particular, at order 1 in the new

couplings ξ^α , (5.22) implies that $\mu_{\xi^\alpha=0}^\alpha$ is in the kernel of the map l_2 , $f_{\beta\alpha}^A \mu_{\xi^\alpha=0}^\alpha = 0$. We thus see that the cohomology has become “smaller” through the introduction of the additional couplings because the extended differential encodes higher order cohomological restrictions.

(ii) At first sight, it might seem a little strange to introduce new couplings in order to get information on the renormalization of the theory without these couplings: that it is convenient and extremely useful to do so was already realized in the first papers [8, 9, 10, 11] on the subject: the additional (space-time dependent) couplings in these papers are just the sources of the BRS transformations, and can of course be set to zero after renormalization, if one is only interested in the renormalization of the effective action itself.

(iii) The result (5.22) implies also that the \bar{s} cohomology is contained completely in the solution $S(\xi)$ and can be obtained from it by applying $\frac{\partial^R}{\partial \xi^A} \lambda^A(\xi)$, where the coefficients $\lambda^A(\xi)$ are constrained to be s_{Δ^*} cocycles.

5.2.6 “Quantum” Batalin-Vilkovisky formalism on the classical level

If we define

$$\Delta_c^L = (-)^D f^D(\xi) \frac{\partial^L}{\partial \xi^D}, \quad \Delta_c = \frac{\partial^R}{\partial \xi^D} f^D(\xi) \quad (5.25)$$

on F , the following properties of the quantum Batalin-Vilkovisky formalism hold in F : the operator Δ_c^L is nilpotent,

$$\Delta_c^{L^2} = 0, \quad (5.26)$$

(as a consequence of (5.21) or (5.9).) Furthermore,

$$\Delta_c^L(A(\xi), B(\xi)) = (\Delta_c^L A(\xi), B(\xi)) + (-)^{|A|+1} (A(\xi), \Delta_c^L B(\xi)). \quad (5.27)$$

Similar properties also hold for the right derivation Δ_c .

To the standard solution of the master equation S in \mathcal{F} corresponds in F the solution $S(\xi)$ of the extended master equation

$$\frac{1}{2}(S(\xi), S(\xi)) + \Delta_c S(\xi) = 0, \quad (5.28)$$

(which is just rewriting (5.20) using the definition of Δ_c). Because

$$\bar{s} = (S(\xi), \cdot) + \Delta_c^L, \quad (5.29)$$

the \bar{s} cohomology corresponds to the quantum BRST cohomology σ discussed for instance in [91, 26]. Corollary 4 shows how to compute the “quantum” BRST cohomology out of the standard BRST cohomology and the higher order maps encoded in s_{Δ^*} .

In this analogy, putting $\xi = 0$ corresponds to the classical limit $\hbar \rightarrow 0$ of the quantum Batalin-Vilkovisky formalism.

Note however that (i) the space F is not directly an algebra, because the product of two local functionals is not well defined, contrary to the formal discussion of the quantum Batalin-Vilkovisky formalism, where one assumes the space to be an algebra, (ii) the above “quantum” Batalin-Vilkovisky formalism is purely classical and depends only on the BRST cohomology and the higher order maps of the theory.

5.3 Deformations and stability

We consider now one parameter deformations of the extended master equation (5.28), i.e., in the space $F[t]$ of power series in t with coefficients that belong to F , we want to construct $S_t(\xi) = S(\xi) + tS_1(\xi) + t^2S_2(\xi) + \dots$ such that

$$\frac{1}{2}(S_t(\xi), S_t(\xi)) + \Delta_c S_t(\xi) = 0. \quad (5.30)$$

A deformation $S_t(\xi) = S(\xi) + tS_1(\xi)$ to first order in t , i.e., such that $\frac{1}{2}(S_t(\xi), S_t(\xi)) + \Delta_c S_t(\xi) = O(t^2)$ is called an infinitesimal deformation. The term linear in t of an infinitesimal deformation, $S_1(\xi)$, is a cocycle of the extended BRST differential \bar{s} . If $S_1(\xi)$ is a \bar{s} coboundary, we call the infinitesimal deformation trivial, while the parts of $S_1(\xi)$ corresponding to the \bar{s} cohomology are non trivial.

Theorem 6 *Every infinitesimal deformation of the solution S to the extended master equation can be extended to a complete deformation S_t . This extension is obtained by (i) performing a t dependent anticanonical field-antifield redefinition $z \rightarrow z'$, by (ii) performing a t dependent coupling constant redefinition $\xi \rightarrow \xi'$, which does not affect Δ_c , and (iii) by adding to $S(z', \xi')$ a suitable extension determined by both coupling constant and the field-antifield redefinition and vanishing whenever the latter does.*

Furthermore, the deformed solution considered as a function of the new variables $S_t(z(z', \xi'), \xi(\xi'))$ satisfies the extended master equation in terms of the new variables and the cohomology $H(\bar{s}', F')$ of the differential $\bar{s}' = (S_t, \cdot)_{z'} + \Delta_c^{L'}$ in the space F' of functionals depending on z', ξ' is isomorphic to the cohomology $H(\bar{s}, F)$.

Proof. Equation (5.22) implies that $S_1(\xi) = \bar{s}B + \frac{\partial^R S(\xi)}{\partial \xi^C} \mu^C(\xi)$ with $(\xi_D^* f^D(\xi), \xi_C^* \mu^C(\xi))_\xi = 0$. In other words, $S_1(\xi) = (\tilde{S}, B(\xi) + \xi_C^* \mu^C(\xi))$. In the extended space \mathcal{E} , with $z^\alpha = (\phi^a, \phi_a^*)$, consider the anticanonical transformation

$$\begin{aligned} z'^\alpha &= \exp t(\cdot, B(\xi) + \xi_C^* \mu^C(\xi)) \tilde{z}^\alpha \\ &= z^\alpha + t(z^\alpha, B(\xi)) + O(t^2), \end{aligned} \quad (5.31)$$

$$\begin{aligned} \xi'^A &= \exp t(\cdot, B(\xi) + \xi_C^* \mu^C(\xi)) \tilde{\xi}^A = \exp t(\cdot, \xi_C^* \mu^C(\xi))_\xi \xi^A \\ &= \xi^A + t\mu^A(\xi) + O(t^2), \end{aligned} \quad (5.32)$$

$$\begin{aligned} \xi_A'^* &= \exp t(\cdot, B(\xi) + \xi_C^* \mu^C(\xi)) \tilde{\xi}_A^* \\ &= \xi_A^* - t \frac{\partial^L}{\partial \xi^A} (B(\xi) + \xi_C^* \mu^C(\xi)) + O(t^2). \end{aligned} \quad (5.33)$$

Note that $z' = z'(z, \xi)$, $\xi' = \xi'(\xi)$ and $\xi_A^{*'} = \xi_A^{*'}(z, \xi, \xi^*) = g_A(z, \xi) + \xi_B^* g_A^B(\xi)$, for a function $g_A(z, \xi) = -t \frac{\partial^L B}{\partial \xi^A} + O(t^2)$ determined by (5.33) through both B and μ and a function $g_A^B(\xi) = \delta_A^B - t(-)^{A(B+1)} \frac{\partial^L \mu^B}{\partial \xi^A} + O(t^2)$ determined by (5.33) through μ alone.

The master equation (5.3) holds in any variables, and thus also in terms of the primed variables. If we denote functions in terms of the new variables by a prime, we get $\frac{1}{2}(\tilde{S}', \tilde{S}')_{z', \xi'} = 0$. Because the transformation is anticanonical, we also have

$$\frac{1}{2}(\tilde{S}', \tilde{S}')_{z, \xi} = 0. \quad (5.34)$$

Since

$$\tilde{S}' = S' + g_A f'^A + \xi_B^* g_A^B f'^A, \quad (5.35)$$

equation (5.34) splits into

$$\frac{1}{2}(S' + g_A f'^A, S' + g_A f'^A)_z + \frac{\partial^R}{\partial \xi^D}(S' + g_A f'^A) g_E^D f'^E = 0, \quad (5.36)$$

$$\frac{1}{2}(\xi_B^* g_A^B f'^A, \xi_D^* g_C^D f'^C)_\xi = 0. \quad (5.37)$$

We have

$$\frac{d(S' + g_A f'^A)}{dt} \Big|_{t=0} = (S(\xi), B(\xi)) + \Delta_c^L B + \frac{\partial^R S(\xi)}{\partial \xi^D} \mu^D = S_1(\xi), \quad (5.38)$$

and, because $\xi_E^* \mu^E(\xi)$ is a s_{Δ^*} cocycle, the relation

$$g_C^D f'^C = f^D. \quad (5.39)$$

Indeed, if we consider the above canonical transformation with $B = 0$, i.e., $\exp t(\cdot, \xi^* \mu)_\xi$ alone, $\xi_C^* g_D^C f^D(\xi') = \xi^{*'}_D f^D(\xi') = \exp t(\cdot, \xi^* \mu)_\xi \xi_E^* f^E = \xi_E^* f^E$, because $(\xi_E^* f^E, \xi_G^* \mu^G)_\xi = 0$. This shows the first part of the theorem, with $S_t = S' + g_A f'^A$.

In order to prove the second part, we first note that

$$\Delta_c^{L'} = (-)^D f^D(\xi') \frac{\partial^L}{\partial \xi'^D} = (-)^D f^D(\xi') \frac{\partial^L \xi^C}{\partial \xi'^D} \frac{\partial^L}{\partial \xi^C} = \Delta_c^L \quad (5.40)$$

because

$$g_D^C = \frac{\partial^L \xi^C}{\partial \xi'^D}. \quad (5.41)$$

Indeed, we have $\delta_B^A = (\xi'^A, \xi_B^{*'})_{z', \xi'} = (\xi'^A, \xi_B^{*'})_\xi = \frac{\partial^L \xi'^A}{\partial \xi'^C} g_B^C$. Together with (5.36), this implies

$$\frac{1}{2}(S' + g_A f'^A, S' + g_A f'^A)_{z'} + \Delta'_c(S' + g_A f'^A) = 0. \quad (5.42)$$

We then start from the relations

$$(\tilde{S}', \mathcal{A}')_{z', \xi'} = 0 \iff \mathcal{A}' = (\tilde{S}', \mathcal{B}')_{z', \xi'}, \quad (5.43)$$

where $\mathcal{A}' = A' + \xi^{*'}_A \lambda'^A$ and $\mathcal{B}' = B' + \xi^{*'}_A \rho'^A$. These relations hold with the bracket taken in the old variables, because the transformation is anticanonical. Writing the resulting relations explicitly, using (5.39), we get that the set of relations

$$\begin{cases} (S' + g_A f'^A, A' + g_B \lambda'^B) + \frac{\partial^R}{\partial \xi^C} (S' + g_A f'^A) g_D^C \lambda'^D \\ + (-)^A f^A \frac{\partial^L}{\partial \xi^A} (A' + g_B \lambda'^B) = 0, \\ \xi_A^* \frac{\partial^R g_B^A f'^B}{\partial \xi^C} g_D^C \lambda'^D + (-)^D g_C^D f'^C \frac{\partial^L}{\partial \xi^D} (\xi_A^* g_B^A \lambda'^B) = 0, \end{cases} \quad (5.44)$$

is equivalent to the set

$$\begin{cases} A' + g_B \lambda'^B = (S' + g_A f'^A, B' + g_C \rho'^C) + \frac{\partial^R}{\partial \xi^C} (S' + g_A f'^A) g_D^C \rho'^D \\ + (-)^A f^A \frac{\partial^L}{\partial \xi^A} (B' + g_C \rho'^C), \\ \xi_A^* g_B^A \lambda'^B = \xi_A^* \frac{\partial^R g_B^A f'^B}{\partial \xi^C} g_D^C \rho'^D + (-)^D g_C^D f'^C \frac{\partial^L}{\partial \xi^D} (\xi_A^* g_B^A \rho'^B). \end{cases} \quad (5.45)$$

Using (5.41), the last equations in (5.44) and (5.45) are just the s_{Δ^*} cocycle and coboundary conditions, expressed in terms of the $\xi^{*'}, \xi'$ variables. Following the same reasoning as in the proof of theorem 4, we get,

$$\begin{aligned} (S' + g_A f'^A, A') + \Delta_c^L A' &= 0 \\ \iff A' &= (S' + g_A f'^A, B' + g_C \rho'^C) \\ + \Delta_c^L (B' + g_C \rho'^C) &+ \frac{\partial^R}{\partial \xi'^C} (S' + g_A f'^A) \rho'^C, \end{aligned} \quad (5.46)$$

$$(\xi^{*'}_A f'^A, \xi^{*'}_B \rho'^B)_{\xi'} = 0, \quad (5.47)$$

where $\frac{\partial^R}{\partial \xi'^C} (S' + g_A f'^A) \rho'^C$ is $(S' + g_A f'^A, \cdot) + \Delta_c^L$ exact iff $\xi^{*'}_B \rho'^B = (\xi^{*'}_A f'^A, \xi^{*'}_C \rho'^C)_{\xi'}$. Since $(S' + g_A f'^A, \cdot) + \Delta_c^L = (S' + g_A f'^A, \cdot)_{z'} + \Delta_c^L = \bar{s}'$, we get that $H(\bar{s}', F')$ is determined by $\frac{\partial^R}{\partial \xi'^C} (S' + g_A f'^A) \rho'^C$ corresponding to the class $\frac{\partial^R S}{\partial \xi'^C} \rho^C$ of $H(\bar{s}, F)$. \square

Remark: Note that one can prove in the same way that the relations $\frac{1}{2}(A, A) + \Delta_c A = C$ and $(A, D) + \Delta_c^L D = E$ become, after the change of variables, $\frac{1}{2}(A' + g_A f'^A, A' + g_A f'^A) + \Delta_c(A' + g_A f'^A) = C'$, respectively $(A' + g_A f'^A, D') + \Delta_c^L D' = E'$.

6 Extended antifield formalism. Quantum theory

We show how the renormalization can be performed while respecting the symmetry, encoded in the extended master equation for the starting point action : the corresponding renormalized effective action satisfies a suitably deformed master equation. In other words, the Zinn-Justin equation can be written to all orders as a functional differential equation without breakings.

In a first part, we consider a regularization and discuss the absorption of divergences by field-antifield and coupling constant redefinitions. In a second part, we derive the same results by applying the methods of algebraic renormalization, relying on the use of the renormalized quantum action principles, in the context of the extended antifield formalism.

6.1 Regularization and absorption of divergences

We make the same assumptions on the regularization as in 4.1 and apply it to the extended master equation (5.28) and its solution $S(\xi)$. In the following, we will always understand the ξ dependence without explicitly indicating it. Local functionals are understood to belong to F .

Let $\theta_\tau = \frac{1}{2\tau}(S_\tau, S_\tau) + \frac{1}{\tau}\Delta_c S_\tau$. Note that θ_τ is of order τ^0 because S_0 satisfies the extended master equation. θ_τ characterizes the breaking of the extended master equation due to the regularization. In order to control this breaking during renormalization, it is useful to couple it with a global source ρ^* in ghost number -1 and consider $S_{\rho^*} = S_\tau + \theta_\tau \rho^*$. On the classical, regularized level, we have, using $(\rho^*)^2 = 0$, and the properties (5.26) and (5.27) of Δ_c ,

$$\frac{1}{2}(S_{\rho^*}, S_{\rho^*}) + \Delta_c S_{\rho^*} = \tau \frac{\partial^R S_{\rho^*}}{\partial \rho^*}, \quad (6.1)$$

Applying the quantum action principle, we get, for the regularized generating functional for 1PI irreducible vertex functions Γ_{ρ^*} associated to S_{ρ^*} ,

$$\frac{1}{2}(\Gamma_{\rho^*}, \Gamma_{\rho^*}) + \Delta_c \Gamma_{\rho^*} = \tau \frac{\partial^R \Gamma_{\rho^*}}{\partial \rho^*}, \quad (6.2)$$

which splits, using $(\rho^*)^2 = 0$, into

$$\frac{1}{2}(\Gamma, \Gamma) + \Delta_c \Gamma = \tau \frac{\partial^R \Gamma_{\rho^*}}{\partial \rho^*}, \quad (6.3)$$

$$(\Gamma, \frac{\partial^R \Gamma_{\rho^*}}{\partial \rho^*}) + \Delta_c^L \frac{\partial^R \Gamma_{\rho^*}}{\partial \rho^*} = 0. \quad (6.4)$$

6.1.1 Invariant regularization

Before proceeding with the general analysis, let us briefly discuss the case when the regularization respect the symmetries under consideration. In this case, θ_τ vanishes

and we have

$$\frac{1}{2}(S_\tau, S_\tau) + \Delta_c S_\tau = 0. \quad (6.5)$$

For the regularized generating functional, we get

$$\frac{1}{2}(\Gamma, \Gamma) + \Delta_c \Gamma = 0, \quad (6.6)$$

where by assumption, $\Gamma = S_\tau + \hbar \sum_{n=-1} \tau^n \Gamma^{(1)n} + O(\hbar^2)$. To order \hbar/τ , (6.6) gives

$$\bar{s}\Gamma^{(1)-1} = 0 \iff \Gamma^{(1)-1} = \bar{s}\Xi_1 + \frac{\partial^R S_0}{\partial \xi^A} \mu_1^A, \quad (6.7)$$

where $\bar{s} = (S_0, \cdot) + \Delta_c^L$ and $s_{\Delta^*} \xi_A^* \mu_1^A = 0$.

We then make the following change of fields, antifields and coupling constants:

$$z^1 = \exp -\frac{\hbar}{\tau}(\cdot, \Xi_1 + \xi^* \mu_1) \tilde{z}, \quad (6.8)$$

$$\xi^1 = \exp -\frac{\hbar}{\tau}(\cdot, \xi^* \mu_1)_\xi \xi. \quad (6.9)$$

If we denote by a superscript 1 functions depending on these new variables, we have, according to the remark after theorem 6, that the action $S_{R1} = S_\tau^1 + g_{1A} f^{1A}$, with g_{1A} is determined through the generators Ξ_1 and μ_1^C of the first redefinition, satisfies the extended master equation (5.28),

$$\frac{1}{2}(S_{R1}, S_{R1}) + \Delta_c S_{R1} = 0. \quad (6.10)$$

It allows to absorb the one loop divergences, since $S_{R1} = S_\tau - \hbar/\tau \Gamma^{(1)-1} + \hbar O(\tau^0) + O(\hbar^2)$. We thus have for the corresponding regularized generating functional $\Gamma_{R1} = S_\tau + \hbar \sum_{n=0} \tau^n \Gamma_{R1}^{(1)n} + \hbar^2 \sum_{n=-2} \tau^n \Gamma_{R1}^{(2)n} + O(\hbar^2)$,

$$\frac{1}{2}(\Gamma_{R1}, \Gamma_{R1}) + \Delta_c \Gamma_{R1} = 0. \quad (6.11)$$

At order \hbar^2/τ^2 , we get

$$\bar{s}\Gamma_{R1}^{(2)-2} = 0 \iff \Gamma^{(2)-2} = \bar{s}\Xi_{2,-2} + \frac{\partial^R S_0}{\partial \xi^A} \mu_{2,-2}^A, \quad (6.12)$$

with $s_{\Delta^*} \xi_A^* \mu_{2,-2}^A = 0$. The appropriate change of variables is

$$z^{2,-2} = \exp -\frac{\hbar^2}{\tau^2}(\cdot, \Xi_{2,-2} + \xi^* \mu_{2,-2}) \tilde{z}, \quad (6.13)$$

$$\xi^{2,-2} = \exp -\frac{\hbar^2}{\tau^2}(\cdot, \xi^* \mu_{2,-2})_\xi \xi. \quad (6.14)$$

The regularized action $S_{R_{2,-2}} = S_{R_1}^{2,-2} + g_{2,-2A} f^{2,-2A}$ satisfies the extended master equation and allows to absorb the poles of order \hbar^2/τ^2 :

$$\frac{1}{2}(S_{R_{2,-2}}, S_{R_{2,-2}}) + \Delta_c S_{R_{2,-2}} = 0, \quad (6.15)$$

and $\Gamma_{R_{2,-2}} = S_\tau + \hbar \sum_{n=0} \Gamma_{R_{2,-2}}^{(1)n} + \hbar^2 \sum_{n=-1} \tau^n \Gamma_{R_{2,-2}}^{(2)n} + O(\hbar^3)$.

In the same way, one can then proceed to absorb the poles of order \hbar^2/τ to get a regularized action $S_{R_{2,-1}}$ and an associated two loop finite effective action $\Gamma_{R_{2,-1}}$, with both actions satisfying the extended master equation.

Going on recursively to higher orders in \hbar , we can achieve, through a succession of redefinitions, the absorptions of the infinities to arbitrary high order in the loop expansion, while preserving the extended master equation for the redefined action and the corresponding generating functional,

$$\frac{1}{2}(S_{R_\infty}, S_{R_\infty}) + \Delta_c S_{R_\infty} = 0, \quad (6.16)$$

with Γ_{R_∞} finite and satisfying

$$\frac{1}{2}(\Gamma_{R_\infty}, \Gamma_{R_\infty}) + \Delta_c \Gamma_{R_\infty} = 0. \quad (6.17)$$

We have thus shown:

Theorem 7 *In theories admitting an invariant regularization scheme, the divergences can be absorbed by successive redefinitions in such a way that both the subtracted and the effective action satisfy the extended master equation.*

6.1.2 Structural constraints and cohomology of \bar{s}

Structural constraints have been introduced in [42] to give in particular cases a sufficient, but not a necessary condition for renormalizability in the modern sense. An example of a structural constraint is the requirement that in every BRST cohomological class in ghost number 0, there exists a representative that is independent of the antifields. In the cases of semi-simple Yang-Mills theories or gravity for instance, this constraint is fulfilled. It guarantees that one can couple these representatives to the action in such a way that the extended action satisfies the same, unmodified master equation. In non anomalous theories, the infinities can then be absorbed by successive coupling constant and field-antifield redefinitions in such a way that the standard master equation holds, both for the subtracted action and the effective action.

What has been shown in the previous section is that structural constraints are not necessary conditions for renormalizability in the modern sense. If one uses the extended antifield formalism, renormalizability in the modern sense can be proved independently of any structural constraint. This is because the extended antifield formalism is stable by construction, due to the fact that the cohomology of the operator \bar{s} incorporates higher order cohomological restrictions.

In the case where the regularization respects the extended master equation, the extended master equation for the effective action implies the following stability of the quantum theory: while the expression of the generalized observables of the theory are affected by quantum corrections, their antibracket algebra stays the same than in the classical theory. In particular, the usual algebra of the generators of the global symmetries (whether linear or not) is the same in the classical and the quantum theory⁶. This is because the antibracket algebra of the BRST cohomology classes in negative ghost numbers just reflects the ordinary algebra of the symmetries they represent.

6.1.3 One loop divergences and anomalies

Let us now go back to the general case where the regularization scheme is not invariant and θ_τ does not vanish.

At one loop, we get from (6.3) and (6.4)

$$(S_\tau, \Gamma^{(1)}) + \Delta_c^L \Gamma^{(1)} = \tau \theta^{(1)}, \quad (6.18)$$

$$(S_\tau, \theta^{(1)}) + (\Gamma^{(1)}, \theta_\tau) + \Delta_c^L \theta^{(1)} = 0, \quad (6.19)$$

where $\Gamma^{(1)}$ and $\theta^{(1)}$ are respectively the one loop contributions of Γ and $\frac{\partial^R \Gamma}{\partial \rho^*}$. By assumption, we have both $\Gamma^{(1)} = \sum_{n=-1} \tau^n \Gamma^{(1)n}$ and $\theta^{(1)} = \sum_{n=-1} \tau^n \theta^{(1)n}$, where $\Gamma^{(1)-1}, \theta^{(1)-1}$ are local functionals.

At $\frac{1}{\tau}$, equation (6.18) gives

$$\bar{s} \Gamma^{(1)-1} = 0, \quad (6.20)$$

Using this equation together with $\theta_0 = \bar{s} S_1$, equation (6.19) implies

$$\bar{s}(\theta^{(1)-1} - (\Gamma^{(1)-1}, S_1)) = 0. \quad (6.21)$$

Equation (6.18) also gives at order τ^0

$$\bar{s} \Gamma^{(1)0} = \theta^{(1)-1} - (\Gamma^{(1)-1}, S_1), \quad (6.22)$$

which allows us to identify the combination $A_1 = \theta^{(1)-1} - (\Gamma^{(1)-1}, S_1)$ as the one loop anomaly and explicitly shows its locality. We have thus shown in the case of a non invariant regularization scheme:

Theorem 8 *The one loop divergences $\Gamma^{(1)-1}$ and the one loop anomalies A_1 are \bar{s} cocycles in ghost number 0 and 1 respectively.*

⁶The author is grateful to F. Brandt for pointing this out.

6.1.4 One loop renormalization

According to (5.22), we have

$$\Gamma^{(1)-1} = \bar{s}\Xi_1 + \frac{\partial^R S_0}{\partial \xi^D} \mu_1^D \quad (6.23)$$

and

$$A_1 = \bar{s}\Sigma_1 + \frac{\partial^R S_0}{\partial \xi^E} \sigma_1^E, \quad (6.24)$$

with $s_{\Delta^*} \xi_A^* \mu_1^A = 0 = s_{\Delta^*} \xi_B^* \sigma_1^B$. The appropriate change of variables is now

$$z^1 = \exp -\frac{\hbar}{\tau}(\cdot, \Xi_1 + \xi^* \mu_1) \tilde{z}, \quad (6.25)$$

$$\xi^1 = \exp -\frac{\hbar}{\tau}(\cdot, \xi^* \mu_1) \tilde{\xi}. \quad (6.26)$$

The renormalized one loop action is

$$S_{R_1} = S_\tau^1 + g_{1A} f^{1A} - \hbar \tilde{\Sigma}_1^1 = S_\tau - \frac{\hbar}{\tau} \Gamma^{(1)-1} + \hbar O(\tau^0) + O(\hbar^2), \quad (6.27)$$

where $\tilde{\Sigma}_1$ remains to be determined. Using the remark after theorem 6, we get

$$\begin{aligned} \theta_{R_1} &\equiv \frac{1}{2\tau}(S_{R_1}, S_{R_1}) + \frac{1}{\tau} \Delta_c S_{R_1} = \theta_\tau^1 - \frac{\hbar}{\tau} (\bar{s}\tilde{\Sigma}_1)^1 + O(\hbar^2) \\ &= \theta_\tau - \frac{\hbar}{\tau} \bar{s}[\tilde{\Sigma}_1 + (S_1, \Xi_1) + \frac{\partial^R S_1}{\partial \xi^A} \mu_1^A] - \frac{\hbar}{\tau} (\Gamma^{(1)-1}, S_1) + \hbar O(\tau^0) + O(\hbar^2). \end{aligned} \quad (6.28)$$

Finally, we consider $\xi_{\rho^*}^1 = \exp -\frac{\hbar}{\tau}(\cdot, \xi^* \sigma_1 \rho^*) \xi = \xi - \frac{\hbar}{\tau} \sigma_1 \rho^*$ and substitute ξ by $\xi_{\rho^*}^1$:

$$\begin{aligned} S_{R_1}^{\rho^*}(z, \xi, \rho^*) &\equiv S_{R_1}(z, \xi_{\rho^*}^1(\xi, \rho^*)) \\ &= S_{R_1}(z, \xi) - \frac{\hbar}{\tau} \frac{\partial^R S_{R_1}}{\partial \xi^A} \sigma_1^A \rho^* \end{aligned} \quad (6.29)$$

$$= S_{R_1}(z, \xi) - \frac{\hbar}{\tau} \frac{\partial^R S_0}{\partial \xi^A} \sigma_1^A \rho^* + \hbar O(\tau^0) + O(\hbar^2). \quad (6.30)$$

We also have that

$$\begin{aligned} \theta_{R_1}^{\rho^*}(z, \xi, \rho^*) &\equiv \theta_{R_1}(z, \xi_{\rho^*}^1(\xi, \rho^*)) \\ &= \theta_{R_1}(z, \xi) - \frac{\hbar}{\tau} \frac{\partial^R \theta_{R_1}}{\partial \xi^A} \sigma_1^A \rho^* \end{aligned} \quad (6.31)$$

$$= \frac{1}{2\tau}(S_{R_1}^{\rho^*}, S_{R_1}^{\rho^*}) + \frac{1}{\tau} \Delta_c S_{R_1}^{\rho^*}. \quad (6.32)$$

Equations (6.28) and (6.30) imply that the action

$$S_{R_1 \rho^*} = S_{R_1}^{\rho^*} + \theta_{R_1}^{\rho^*} \rho^*, \quad (6.33)$$

with $\tilde{\Sigma}_1 = \Sigma_1 - (S_1, \Xi_1) - \frac{\partial^R S_1}{\partial \xi^A} \mu_1^A$, yields a one loop finite effective action both in the ρ^* independent and the ρ^* linear part, because the terms linear in ρ^* of order \hbar/τ add up precisely to $-\theta^{(1)-1}$. The one loop renormalized and regularized action $S_{R_1\rho^*}$ satisfies

$$\begin{aligned} & \frac{1}{2}(S_{R_1\rho^*}, S_{R_1\rho^*}) + \Delta_c S_{R_1\rho^*} = \tau \theta_{R_1}^{\rho^*} \\ & = \tau \frac{\partial^R S_{R_1\rho^*}}{\partial \rho^*} + \frac{\partial^R S_{R_1\rho^*}}{\partial \xi^B} \hbar \sigma_1^B - \frac{1}{\tau} \frac{\partial^R S_{R_1\rho^*}}{\partial \xi^B} \frac{\partial^R \hbar \sigma_1^B}{\partial \xi^A} \hbar \sigma_1^A \rho^*, \end{aligned} \quad (6.34)$$

the first equality following from (6.32), and the last equality from the expansions (6.29), (6.31), together with the identity

$$(-)^{B(A+1)} \frac{\partial^R}{\partial \xi^B} \left(\frac{\partial^R S_{R_1}}{\partial \xi^A} \right) \sigma_1^A \sigma_1^B = 0. \quad (6.35)$$

Let us now define $\Delta^1 = \Delta_c - \hbar \frac{\partial^R}{\partial \xi^A} \sigma_1^A$, with $\frac{\partial^R}{\partial \xi^B} [\Delta^1, \Delta^1]^B = O(\hbar^2)$. We can then write

$$\frac{1}{2}(S_{R_1\rho^*}, S_{R_1\rho^*}) + \Delta^1 S_{R_1\rho^*} = \tau \frac{\partial^R S_{R_1\rho^*}}{\partial \rho^*} - \frac{1}{\tau} \frac{\partial^R S_{R_1\rho^*}}{\partial \xi^B} [\Delta^1, \Delta^1]^B \rho^*, \quad (6.36)$$

According to the regularized quantum action principle,

$$\frac{1}{2}(\Gamma_{R_1\rho^*}, \Gamma_{R_1\rho^*}) + \Delta^1 \Gamma_{R_1\rho^*} = \tau \frac{\partial^R \Gamma_{R_1\rho^*}}{\partial \rho^*} - \frac{1}{\tau} \frac{\partial^R \Gamma_{R_1\rho^*}}{\partial \xi^B} [\Delta^1, \Delta^1]^B \rho^*. \quad (6.37)$$

The ρ^* independent part at one loop and lowest order, τ^0 , in τ gives

$$\bar{s} \Gamma_{R_1}^{(1)0} = \frac{\partial^R S_0}{\partial \xi^B} \sigma_1^B, \quad (6.38)$$

and shows that only the non trivial part of the anomaly remains.

6.1.5 Two loops

Equations for the two loop poles The one loop renormalized action admits the expansion

$$\Gamma_{R_1\rho^*} = S_{\rho^*} + \hbar \Sigma_{n=0} \tau^n \Gamma_{R_1\rho^*}^{(1)n} + \hbar^2 \Sigma_{n=-2} \tau^n \Gamma_{R_1\rho^*}^{(2)n} + O(\hbar^3). \quad (6.39)$$

At order \hbar^2 , (6.37) gives

$$\begin{aligned} & (S_{\rho^*}, \Gamma_{R_1\rho^*}^{(2)}) + \frac{1}{2}(\Gamma_{R_1\rho^*}^{(1)}, \Gamma_{R_1\rho^*}^{(1)}) + \Delta_c \Gamma_{R_1\rho^*}^{(2)} \\ & = \tau \frac{\partial^R \Gamma_{R_1\rho^*}^{(2)}}{\partial \rho^*} + \frac{\partial^R \Gamma_{R_1\rho^*}^{(1)}}{\partial \xi^B} \sigma_1^B - \frac{1}{\tau} \frac{\partial^R S_0}{\partial \xi^B} \frac{\partial^R \sigma_1^B}{\partial \xi^A} \sigma_1^A \rho^*. \end{aligned} \quad (6.40)$$

Let $\Gamma_{R_1\rho^*} = \Gamma_{R_1} + \frac{\partial^R \Gamma_{R_1\rho^*}}{\partial \rho^*} \rho^*$. At order $1/\tau^2$, we get, according to the ρ^* independent and linear parts,

$$\bar{s}\Gamma_{R_1}^{(2)-2} = 0, \quad (6.41)$$

$$\bar{s}\left(\frac{\partial^R \Gamma_{R_1\rho^*}^{(2)-2}}{\partial \rho^*} - (S_1, \Gamma_{R_1}^{(2)-2})\right) = 0. \quad (6.42)$$

The first of these equations implies:

Lemma 1 *The second order pole of the two loop divergences is a \bar{s} cocycle.*

At order $1/\tau$, we get

$$\bar{s}\Gamma_{R_1}^{(2)-1} = \frac{\partial^R \Gamma_{R_1\rho^*}^{(2)-2}}{\partial \rho^*} - (S_1, \Gamma_{R_1}^{(2)-2}), \quad (6.43)$$

$$\bar{s}\left(\frac{\partial^R \Gamma_{R_1\rho^*}^{(2)-1}}{\partial \rho^*} - (S_1, \Gamma_{R_1}^{(2)-1}) - (S_2, \Gamma_{R_1}^{(2)-2})\right) = -\frac{\partial^R S_0}{\partial \xi^B} \frac{\partial^R \sigma_1^B}{\partial \xi^A} \sigma_1^A. \quad (6.44)$$

Finally, the ρ^* independent part of (6.40), gives at order τ^0

$$\begin{aligned} \bar{s}\Gamma_{R_1}^{(2)0} + \frac{1}{2}(\Gamma_{R_1}^{(1)0}, \Gamma_{R_1}^{(1)0}) &= \frac{\partial^R \Gamma_{R_1}^{(1)0}}{\partial \xi^B} \sigma_1^B \\ &+ \frac{\partial^R \Gamma_{R_1\rho^*}^{(2)-1}}{\partial \rho^*} - (S_1, \Gamma_{R_1}^{(2)-1}) - (S_2, \Gamma_{R_1}^{(2)-2}), \end{aligned} \quad (6.45)$$

which allows to identify the combination

$$A_2 = \frac{\partial^R \Gamma_{R_1\rho^*}^{(2)-1}}{\partial \rho^*} - (S_1, \Gamma_{R_1}^{(2)-1}) - (S_2, \Gamma_{R_1}^{(2)-2}) \quad (6.46)$$

as the local contribution to the two loop anomaly, whereas $\frac{\partial^R \Gamma_{R_1}^{(1)0}}{\partial \xi^B} \sigma_1^B$ is the one loop renormalized dressing of the non trivial one loop anomaly.

Two loop anomaly consistency condition Before absorbing the divergences, let us consider (6.44), which can be written as

$$\bar{s}A_2 = -\frac{1}{2} \frac{\partial^R S_0}{\partial \xi^B} [\sigma_1, \sigma_1]^B, \quad (6.47)$$

where $\xi_B^*[\sigma_1, \sigma_1]^B \equiv (\xi^* \sigma_1, \xi^* \sigma_1)_\xi$ is an s_{Δ^*} cocycle because of the graded Jacobi identity for the antibracket in ξ, ξ^* space. According (5.23), this implies that

$$\frac{1}{2}(\xi^* \sigma_1, \xi^* \sigma_1)_\xi = s_{\Delta^*} \xi_A^* \sigma_2^A, \quad (6.48)$$

and

$$A_2 = \bar{s}\Sigma_2 + \frac{\partial^R S_0}{\partial \xi^B} \sigma_2^B. \quad (6.49)$$

Discussion: We thus see that the consistency condition (6.47) on the local contribution of the two loop anomaly does not require it to be just a cocycle of the extended BRST differential \bar{s} , because of the non vanishing right hand side. This is in agreement with the analysis of [52, 53]. Nevertheless, in the extended anti-field formalism, the general solution (6.49) of (6.47) can be given. Up to a trivial \bar{s} boundary, it contains the term $\frac{\partial^R S_0}{\partial \xi^B} \sigma_2^B$ with the following interpretation. From the point of view of cohomology, equation (6.47) should be understood as a restriction on the non trivial one loop anomalies $\xi_A^* \sigma_1^A$ that can arise. Indeed, its consequence is (6.48), which states that the non trivial one loop anomalies should have a trivial antibracket map⁷ among themselves. This is a cohomological statement independent of the choice of representatives. Through these cohomological considerations, the term $\frac{\partial^R S_0}{\partial \xi^B} \sigma_2^B$ of the general solution for the local part of the two loop anomaly is determined up to an arbitrary s_{Δ^*} cocycle, (or, using the liberty to shift the s_{Δ^*} trivial part of it in the \bar{s} coboundary, up to a s_{Δ^*} cohomological class). It contains a particular solution depending on the choice of representatives for the non trivial one loop anomalies and needed to make the bracket $(\xi^* \sigma_1, \xi^* \sigma_1)_\xi$ s_{Δ^*} exact. This answers, at least in the present context of the extended antifield formalism the question raised in [52, 53] on the cohomological interpretation of the two loop anomaly consistency condition. It is confirmed by the analysis in the next subsection in the context of algebraic renormalization. One also sees on this example how the discussion of the quantum Batalin-Vilkovisky formalism of [52, 53] is shifted to ξ, ξ^* space in the extended formalism.

Note that, as in [92], this result has been achieved by adding a BRST breaking counterterm, not only for the one loop divergences produced by the standard action itself, but also for the one loop divergences produced by the insertion of the non trivial one loop anomaly. This is because this anomaly has been coupled to the action itself from the start, and the BRST breaking counterterm Σ_1 also depends on the corresponding coupling constants.

Two loop renormalization The general solution to (6.41) is $\Gamma_{R_1}^{(2)-2} = \bar{s} \Xi_{2,-2} + \frac{\partial^R S_0}{\partial \xi^A} \mu_{2,-2}^A$. We consider the change of variables

$$z^{2,-2} = \exp - \frac{\hbar^2}{\tau^2} [(\cdot, \Xi_{2,-2}^{\rho^*}) + (\cdot, \xi_{\rho^*}^{1*} \mu_{2,-2}^{\rho^*})_{\xi_{\rho^*}^1}] z, \quad (6.50)$$

$$\xi^{2,-2} = \exp - \frac{\hbar^2}{\tau^2} (\cdot, \xi_{\rho^*}^{1*} \mu_{2,-2}^{\rho^*})_{\xi_{\rho^*}^1} \xi_{\rho^*}^1, \quad (6.51)$$

where $\Xi_{2,-2}^{\rho^*}(z, \xi, \rho^*) = \Xi_{2,-2}(z, \xi_{\rho^*}^1(\xi, \rho^*))$ and $\mu_{2,-2}^{\rho^*}(\xi, \rho^*) = \mu_{2,-2}(\xi_{\rho^*}^1(\xi, \rho^*))$. The fact that we consider this change of variables in terms of $\xi_{\rho^*}^1$ instead of ξ will not change the absorption of the ρ^* independent divergences, but it will be important in order to control the dependence on ρ^* below. Equation (6.43) means that there is no non trivial part $\frac{\partial^R S_0}{\partial \xi^A} \sigma_{2,-2}^A$ in the general solution to (6.42) and hence no need for

⁷The antibracket map here is the antibracket induced in the s_{Δ^*} cohomological classes from the antibracket in ξ space

a renormalization of the coupling constants of order \hbar^2/τ^2 proportional to ρ^* . The general solution to (6.42) is $\frac{\partial^R \Gamma_{R_1 \rho^*}^{(2)-2}}{\partial \rho^*} - (S_1, \Gamma_{R_1}^{(2)-2}) = \bar{s} \Sigma_{2,-2}$, where $\Sigma_{2,-2}$ can be identified with a particular solution $\Gamma_{R_1 P}^{(2)-1}$ of (6.43). We take

$$\begin{aligned} S_{R_{2,-2}}^{\rho^*}(z, \xi, \rho^*) &= S_{R_1}(z^{2,-2}(z, \xi_{\rho^*}^1), \xi^{2,-2}(\xi_{\rho^*}^1)) + g_{2,-2A}(z, \xi_{\rho^*}^1) f^A(\xi_{\rho^*}^1) \\ &\quad - \frac{\hbar^2}{\tau} \tilde{\Sigma}_{2,-2}(z^{2,-2}(z, \xi_{\rho^*}^1), \xi^{2,-2}(\xi_{\rho^*}^1)), \\ &= S_{R_1} - \frac{\hbar^2}{\tau^2} \Gamma_{R_1}^{(2)-2} + O(\hbar^2 \tau^{-1}) + O(\hbar^3), \end{aligned} \quad (6.52)$$

where $\tilde{\Sigma}_{2,-2}(z, \xi)$ remains to be determined. The remark after theorem 6 again implies

$$\begin{aligned} \theta_{R_{2,-2}}^{\rho^*} &\equiv \frac{1}{2\tau} (S_{R_{2,-2}}^{\rho^*}, S_{R_{2,-2}}^{\rho^*}) + \frac{1}{\tau} \Delta_c S_{R_{2,-2}}^{\rho^*} \\ &= \theta_{R_1}^{\rho^*} - \frac{\hbar^2}{\tau^2} \bar{s} [\tilde{\Sigma}_{2,-2} + \frac{\partial^R S_1}{\partial \xi^A} \mu_{2,-2}^A + (S_1, \Xi_{2,-2})] \\ &\quad - \frac{\hbar^2}{\tau^2} (S_1, \Gamma_{R_1}^{(2)-2}) + \hbar^2 O(\tau^{-1}) + O(\hbar^3). \end{aligned} \quad (6.53)$$

The action

$$S_{R_{2,-2} \rho^*} = S_{R_{2,-2}}^{\rho^*} + \theta_{R_{2,-2}}^{\rho^*} \rho^*, \quad (6.54)$$

with $\tilde{\Sigma}_{2,-2} = \Sigma_{2,-2} - \frac{\partial^R S_1}{\partial \xi^A} \mu_{2,-2}^A + (S_1, \Xi_{2,-2})$, yields an effective action $\Gamma_{R_{2,-2} \rho^*}$ without \hbar^2/τ^2 divergences and only simple poles at order \hbar^2 , because the terms linear in ρ^* of order \hbar^2/τ^2 add up precisely to $-\frac{\partial^R \Gamma_{R_1 \rho^*}^{(2)-2}}{\partial \rho^*}$. We have again that

$$\begin{aligned} &\frac{1}{2} (S_{R_{2,-2} \rho^*}, S_{R_{2,-2} \rho^*}) + \Delta_c S_{R_{2,-2} \rho^*} = \tau \theta_{R_{2,-2}}^{\rho^*} \\ &= \tau \frac{\partial^R S_{R_{2,-2} \rho^*}}{\partial \rho^*} + \frac{\partial^R S_{R_{2,-2} \rho^*}}{\partial \xi^B} \hbar \sigma_1^B - \frac{1}{\tau} \frac{\partial^R S_{R_{2,-2} \rho^*}}{\partial \xi^B} \frac{1}{2} [\hbar \sigma_1, \hbar \sigma_1]^B \rho^*. \end{aligned} \quad (6.55)$$

The last equation follows from the fact that the dependence of $S_{R_{2,-2}}^{\rho^*}$ and $\theta_{R_{2,-2}}^{\rho^*}$ on ρ^* is, as before, through the combination $\xi_{\rho^*}^1$. The same equation holds again for the effective action:

$$\begin{aligned} \frac{1}{2} (\Gamma_{R_{2,-2} \rho^*}, \Gamma_{R_{2,-2} \rho^*}) + \Delta_c \Gamma_{R_{2,-2} \rho^*} &= \tau \frac{\partial^R \Gamma_{R_{2,-2} \rho^*}}{\partial \rho^*} + \frac{\partial^R \Gamma_{R_{2,-2} \rho^*}}{\partial \xi^B} \hbar \sigma_1^B \\ &\quad - \frac{1}{\tau} \frac{\partial^R \Gamma_{R_{2,-2} \rho^*}}{\partial \xi^B} \frac{1}{2} [\hbar \sigma_1, \hbar \sigma_1]^B \rho^*. \end{aligned} \quad (6.56)$$

The expansion of this effective action is

$$\Gamma_{R_{2,-2} \rho^*} = S_{\rho^*} + \hbar \Sigma_{n=0} \tau^n \Gamma_{R_{2,-2} \rho^*}^{(1)n} + \hbar^2 \Sigma_{n=-1} \tau^n \Gamma_{R_{2,-2} \rho^*}^{(2)n} + O(\hbar^3). \quad (6.57)$$

The divergences $\Gamma_{R_{2,-2}}^{(2)-1}$ and $\frac{\partial^R \Gamma_{R_{2,-2}}^{(2)-1}}{\partial \rho^*}$, now satisfy $\bar{s}\Gamma_{R_{2,-2}}^{(2)-1} = 0$ and $\bar{s}A'_2 = \frac{\partial^R S_0}{\partial \xi^A} \frac{1}{2}[\sigma^1, \sigma^1]^A$, with $A'_2 = \frac{\partial^R \Gamma_{R_{2,-2}}^{(2)-1}}{\partial \rho^*} - (\Gamma_{R_{2,-2}}^{(2)-1}, S_1)$. The general solutions are $\Gamma_{R_{2,-2}}^{(2)-1} = \bar{s}\Xi_{2,-1} + \frac{\partial^R S_0}{\partial \xi^A} \mu_{2,-1}$ and $A'_2 = \bar{s}\Sigma_{2,-1} + \frac{\partial^R S_0}{\partial \xi^A} \sigma_2^A$. As in the one loop case, one first subtracts a suitably defined BRST breaking counterterm, then one makes the field-antifield and coupling constant redefinition determined by $\Xi_{2,-1}^{\rho^*}$ and $\mu_{2,-1}^{\rho^*}$, and finally, one substitutes $\xi_{\rho^*}^1$ everywhere by $\xi_{\rho^*}^2 = \xi_{\rho^*}^1 - \frac{\hbar^2}{\tau} \sigma_2 \rho^*$, giving a total ρ^* dependence through the combination $\xi_{\rho^*}^2 = \xi - \frac{\hbar}{\tau} \sigma_1 \rho^* - \frac{\hbar^2}{\tau} \sigma_2 \rho^*$.

Using the same arguments as in the one loop case, one finally finds⁸ that the two loop renormalized and regularized action $S_{R_{2\rho^*}}$ satisfies, by defining $\Delta^2 = \Delta_c - \hbar \frac{\partial^R}{\partial \xi^A} \sigma_1^A - \hbar^2 \frac{\partial^R}{\partial \xi^A} \sigma_2^A$, with $\frac{\partial^R}{\partial \xi^B} [\Delta^2, \Delta^2]^B = 0(\hbar^3)$,

$$\frac{1}{2}(S_{R_{2\rho^*}}, S_{R_{2\rho^*}}) + \Delta^2 S_{R_{2\rho^*}} = \tau \frac{\partial^R S_{R_{2\rho^*}}}{\partial \rho^*} - \frac{1}{\tau} \frac{\partial^R S_{R_{2\rho^*}}}{\partial \xi^B} \frac{1}{2} [\Delta^2, \Delta^2]^B \rho^*, \quad (6.58)$$

the same equation holding for the two loop renormalized effective action $\Gamma_{R_{2\rho^*}}$.

6.1.6 Higher orders

It is then possible to continue recursively to higher loops to get a completely subtracted and regularized action S_{R_∞} . It is obtained from

$$S_\tau - \sum_{n=1} \frac{\hbar^n}{\tau^{n-1}} \sum_{k=0}^{n-1} \tau^k \tilde{\Sigma}_{n,k-n}, \quad (6.59)$$

with suitably chosen BRST breaking counterterms $\tilde{\Sigma}_{n,k-n}$, by successive canonical field-antifield and coupling constants redefinitions. It satisfies

$$\frac{1}{2}(S_{R_\infty \rho^*}, S_{R_\infty \rho^*}) + \Delta^\infty S_{R_\infty \rho^*} = \tau \frac{\partial^R S_{R_\infty \rho^*}}{\partial \rho^*}, \quad (6.60)$$

with

$$\Delta^\infty = \Delta_c - \sum_{n=1} \hbar \frac{\partial^R}{\partial \xi^A} \sigma_n^A, \quad (6.61)$$

$$\frac{\partial^R}{\partial \xi^B} [\Delta^\infty, \Delta^\infty]^B \equiv (\Delta^\infty)^2 = 0. \quad (6.62)$$

The corresponding completely renormalized and regularized effective action $\Gamma_{R_\infty \rho^*}$ satisfies the same equation.

$$\frac{1}{2}(\Gamma_{R_\infty \rho^*}, \Gamma_{R_\infty \rho^*}) + \Delta^\infty \Gamma_{R_\infty \rho^*} = \tau \frac{\partial^R \Gamma_{R_\infty \rho^*}}{\partial \rho^*}. \quad (6.63)$$

⁸Note that the following equations in this and the next two subsection have been corrected with respect to the ones in the original paper [2] and that theorem 9 below has been accordingly improved.

One can then take safely the limit $\tau \rightarrow 0$, because there are no more divergences left and put ρ^* to zero. The renormalized effective action $\Gamma^\infty = (\lim_{\tau \rightarrow 0} \Gamma_{R_\infty \rho^*})|_{\rho^*=0}$ satisfies

$$\frac{1}{2}(\Gamma^\infty, \Gamma^\infty) + \Delta^\infty \Gamma^\infty = 0. \quad (6.64)$$

Theorem 9 *The absorption of the divergences in the extended antifield formalism involves, besides redefinitions of the solution of the extended master equation, determined by anticanonical field-antifield and coupling constant renormalizations, only the subtraction of suitably chosen BRST breaking counterterms. The renormalization can be done in such a way that the effective action satisfies an extended master equation with a differential Δ^∞ that is a deformation of the differential Δ_c of the classical extended master equation. This statement contains the cohomological information on the anomaly consistency condition to all orders.*

6.1.7 The quantum Batalin-Vilkovisky Δ operator

In [49, 52, 53], explicit expression for the Δ operator have been obtained in the context of Pauli-Villars and non local regularization respectively. The aim of this section is to get such an expression in the context of the present “dimensional” renormalization. The expression we will get here will be defined on all the generalized observables of the theory, and not only on S alone, since they are contained in the solution $S(\xi)$ of the extended master equation.

As discussed for instance in section 4 of [51] in the context of the BPHZ renormalized antifield formalism, even though there is a well defined expression for the anomaly, there is no room for the formal Batalin-Vilkovisky Δ operator in the final renormalized theory. Contact with the quantum Batalin-Vilkovisky formalism in the present set-up has thus to be done on the renormalized theory before the regulator τ is removed. Moreover, as in the previous discussion of the renormalization, it turns out to be important not to put to zero the fermionic variable ρ^* , which couples the breaking of the extended master equation due to the regularization. Let us introduce the notation $W = S_{R_\infty \rho^*}$ for the completely renormalized and regularized action and define $\Delta_d = \frac{\tau}{i\hbar} \frac{\partial R}{\partial \rho^*}$ so that (6.60) becomes

$$\frac{1}{2}(W, W) + (\Delta^\infty - i\hbar\Delta_d)W = 0. \quad (6.65)$$

The operator Δ_d is of ghost number 1, it is nilpotent, $\Delta_d^2 = 0$, it anticommutes with Δ^∞ , $\{\Delta^\infty, \Delta_d\} = 0$, so that $\Delta^T = \Delta^\infty - i\hbar\Delta_d$ is a differential $(\Delta^T)^2 = 0$. Δ_d is also a graded derivation of the antibracket, i.e., it satisfies equation (5.27) (with Δ_c replaced by Δ_d or Δ^T).

Discussion: Starting from the path integral expression

$$Z(J, \phi^*, \xi, \rho^*) = \int \mathcal{D}\phi \exp \left(\frac{i}{\hbar} [W + \int d^n x J_a \phi^a] \right), \quad (6.66)$$

with associated effective action $\Gamma_{R_\infty \rho^*}$, standard formal path integral manipulations using integrations by parts give

$$\frac{1}{2}(\Gamma_{R_\infty \rho^*}, \Gamma_{R_\infty \rho^*}) + \Delta^\infty \Gamma_{R_\infty \rho^*} = \mathcal{A}' \circ \Gamma_{R_\infty \rho^*}, \quad (6.67)$$

where

$$\mathcal{A}' = \frac{1}{2}(W, W) + \Delta^\infty W - i\hbar'' \Delta W''. \quad (6.68)$$

This expression involves the second order functional derivative operator $\Delta = (-)^{A+1} \frac{\delta^R}{\delta \phi^a(x)} \frac{\delta^R}{\delta \phi^*(x)}$. The quotation marks mean that the above definition of Δ cannot be used since Δ is ill defined when acting on local functionals and thus on W . Using (6.63) for the left hand side, we get $\mathcal{A}' = i\hbar \Delta_d W$. Using furthermore (6.65), it follows that $-i\hbar'' \Delta W'' = 0$, as was to be expected in “dimensional” regularization, where $''\delta(0)'' = 0$.

In equation (6.65), obtained by an analysis of the renormalization procedure, there appears the operator Δ_d , which is unexpected from the point of view of formal path integral manipulations, not taking the regularization and renormalization into account. Furthermore, the operator Δ_d has the same algebraic properties as the formal operator Δ , when acting on local functionals. In “dimensional regularization”, one has traded the operator Δ , vanishing on local functionals, for the operator Δ_d . We thus find, in the context of dimensional regularization, that the role of the Batalin-Vilkovisky Δ operator is played by the operator Δ_d , introduced originally in [59].

Furthermore, the terms of higher order in \hbar of Δ^∞ contain the information about the anomalies of the theory, while (6.65) suggests that the operator Δ_c , can be understood as a classical part of the Batalin-Vilkovisky Δ operator. We also note that both Δ_c and Δ_d arise in a similar way from an extended action satisfying a standard master equation in an extended space with an enlarged bracket: this was shown for Δ_c in section 5.2.1. In [65] in the context of the standard Batalin-Vilkovisky formalism, it was shown that Δ_d also arises from an “improved” classical master equation, if the space of fields and antifields is enlarged to include the global pair of variables ρ, ρ^* , the antibracket is extended to this pair and the regularized action is extended to $S_\tau + \theta_\tau \rho^* + \tau \rho$.

6.2 Algebraic renormalization in the extended antifield formalism

The algebraic approach to control symmetries in renormalization theory is based on the use of the renormalized quantum action principles [93, 94, 95, 96, 97], that hold independently of the particular scheme being used.

More precisely, let S_{gf} be the classical gauge fixed action of the theory. If Γ^∞ denotes the renormalized generating functional for one particle irreducible vertices, one has

$$\Gamma^\infty = S_{\text{gf}} + O(\hbar). \quad (6.69)$$

Similarly, if $\Delta \circ \Gamma^\infty$, respectively $\Delta(x) \circ \Gamma^\infty$, denotes the renormalized insertion of an (integrated) local polynomial into Γ , one has

$$\Delta \circ \Gamma^\infty = \Delta + O(\hbar). \quad (6.70)$$

Let g be a parameter of S_{gf} and let $\phi(x)$ be the source for the field operators in the generating functional for one particle irreducible vertex functions with $\rho(x)$ an external source coupling to a polynomial in the fields and their derivatives. We will use the quantum action principle in the following forms:

$$\begin{aligned} & \text{(non linear) field variations :} \\ & \frac{\delta \Gamma^\infty}{\delta \phi(x)} \frac{\delta \Gamma^\infty}{\delta \rho(x)} = \Delta''(x) \circ \Gamma^\infty, \\ & \Delta''(x) \circ \Gamma^\infty = \frac{\delta S_{\text{gf}}}{\delta \phi(x)} \frac{\delta S_{\text{gf}}}{\delta \rho(x)} + O(\hbar). \end{aligned} \quad (6.71)$$

$$\begin{aligned} & \text{coupling constants : } \frac{\partial \Gamma^\infty}{\partial g} = \Delta \circ \Gamma^\infty, \\ & \Delta \circ \Gamma^\infty = \frac{\partial S_{\text{gf}}}{\partial g} + O(\hbar). \end{aligned} \quad (6.72)$$

In the following, we will assume the validity of these equations, even in the context of power counting non renormalizable theories. The precise functional dependence of Γ^∞ on the couplings will not be discussed, we just assume it to be sufficiently regular to allow for the manipulations below.

In the algebraic approach to the usual version of the BRST-Zinn-Justin-Batalin-Vilkovisky set-up, there are two main issues to be considered (see e.g. [46, 47]): stability and anomalies.

6.2.1 Stability

The problem of stability (in the physical sector) is the question if to every local BRST cohomological class $H^{0,n}(s|d)$ in ghost number 0, there corresponds an independent coupling of the standard master equation. If this is the case, every infinitesimal deformation of the action (by invariant, in this context finite, counterterms) can be absorbed by a coupling constant and anticanonical field-antifield redefinition in such a way that the master equation is still satisfied.

The extended formalism solves this problem by construction, because all standard cohomological classes have been coupled with independent couplings. Indeed, in the extended formalism, the differential controlling the “instabilities”, i.e., the divergences and/or counterterms, is the differential \bar{s} . According to theorem 6, every infinitesimal deformation can be absorbed in such a way that the deformed action still satisfies the extended master equation.

Since no additional arguments like power counting have been used to achieve this stability, one can say that the use of the extended antifield formalism guarantees “renormalizability in the modern sense” [42] for all gauge theories.

Of course, it will be often convenient in practice not to couple all the local BRST cohomological classes but only a subset needed to guarantee that the theory is stable, especially if one uses additional conditions like power counting.

6.2.2 Anomalous Zinn-Justin equation

In the standard set-up, the question of anomalies is mostly reduced to the computation of the local BRST cohomological group $H^{1,n}(s|d)$ in ghost number 1 and to a discussion of the coefficients of the corresponding classes. In the presence of anomalies, there is no differential on the quantum level associated to the anomalously broken Zinn-Justin equation for the effective action. In the extended antifield formalism however, because all the local BRST cohomological classes in positive ghost numbers have been coupled to the solution of the master equation, the broken Zinn-Justin equation can be written as a functional differential equation and an associated differential exists, even in the presence of anomalies. To show this, is the object of the remainder of this subsection ⁹.

The quantum action principle applied to (5.28) gives

$$\frac{1}{2}(\Gamma, \Gamma) + \Delta_c \Gamma = \hbar \mathcal{A} \circ \Gamma, \quad (6.73)$$

where Γ is the renormalized generating functional for 1PI vertices associated to the solution S of the extended master equation and the local functional \mathcal{A} is an element of F in ghost number 1. Applying $(\Gamma, \cdot) + \Delta_c^L$ to (6.73), the l.h.s vanishes identically because of the graded Jacobi identity for the antibracket and the properties of Δ_c , so that one gets the consistency condition $(\Gamma, \mathcal{A} \circ \Gamma) + \Delta_c^L \mathcal{A} \circ \Gamma = 0$. To lowest order in \hbar , this gives $\bar{s}\mathcal{A} = 0$, the general solution of which can be written as

$$\mathcal{A} = -\frac{\partial^R S}{\partial \xi^A} \Delta_1^A + \bar{s}\Sigma_1, \quad (6.74)$$

with $[\Delta_c, \Delta_1] = 0$, because of the relation between the \bar{s} and the s_{Δ_c} cohomologies discussed in the previous section.

If one now defines $S^1 = S - \hbar \Sigma_1$, the corresponding generating functional admits the expansion $\Gamma^1 = \Gamma - \hbar \Sigma_1 + O(\hbar^2)$ and satisfies $\frac{1}{2}(\Gamma^1, \Gamma^1) + \Delta^1 \Gamma^1 = O(\hbar^2)$, where $\Delta^1 = \Delta_c + \hbar \Delta_1$. On the other hand, the quantum action principle applied to $\frac{1}{2}(S^1, S^1) + \Delta^1 S^1 = O(\hbar)$ implies $\frac{1}{2}(\Gamma^1, \Gamma^1) + \Delta^1 \Gamma^1 = \hbar \bar{\mathcal{A}} \circ \Gamma^1$, for a local functional $\bar{\mathcal{A}}$. Comparing the two expressions, we deduce that

$$\frac{1}{2}(\Gamma^1, \Gamma^1) + \Delta^1 \Gamma^1 = \hbar^2 \mathcal{A}' \circ \Gamma^1, \quad (6.75)$$

⁹We rederive section 4 of [3] in a more appropriate notation, insisting on the existence of the quantum BRST differential in the anomalous case and its relation to its classical counterpart \bar{s} . Note that the relation after (4.9) of that paper should read $s_Q \frac{\partial^R}{\partial \xi^A} \sigma^A = \frac{1}{2} \frac{\partial^R}{\partial \xi^A} [\sigma, \sigma]^A$ instead of $s_Q \frac{\partial^R}{\partial \xi^A} \sigma^A = 0$.

for a local functional \mathcal{A}' . Applying now $(\Gamma^1, \cdot) + (\Delta^1)^L$, one gets as consistency condition

$$\frac{1}{2}[\Delta^1, \Delta^1]\Gamma^1 + \hbar^2((\Gamma^1, \mathcal{A}' \circ \Gamma^1) + \Delta^1 \mathcal{A}' \circ \Gamma^1) = 0, \quad (6.76)$$

giving to lowest order

$$1/2[\Delta_1, \Delta_1]S + \bar{s}\mathcal{A}' = 0. \quad (6.77)$$

Since $1/2[\Delta_1, \Delta_1]$ is a s_{Δ_c} cocycle because of the graded Jacobi identity for the graded commutator, equation (5.23) imply that the general solution to this equation is

$$\frac{1}{2}[\Delta_1, \Delta_1] + [\Delta_c, \Delta_2] = 0, \quad (6.78)$$

$$\mathcal{A}' = \bar{s}\Sigma_2 - \frac{\partial^R S}{\partial \xi^B} \Delta_2^B. \quad (6.79)$$

The redefinition $S^2 = S^1 - \hbar^2 \Sigma_2$ then allows to achieve

$$\frac{1}{2}(\Gamma^2, \Gamma^2) + \Delta^2 \Gamma^2 = \hbar^3 \mathcal{A}'' \circ \Gamma^2, \quad (6.80)$$

for a local functional \mathcal{A}'' , with $\Delta^2 = \Delta^1 + \hbar^2 \Delta_2$. The reasoning can be pushed recursively to all orders with the result

$$\frac{1}{2}(\Gamma^\infty, \Gamma^\infty) + \Delta^\infty \Gamma^\infty = 0, \quad (6.81)$$

where Γ^∞ is associated to the action $S^\infty = S - \Sigma_{k=1} \hbar^k \Sigma_k$ and $\Delta^\infty = \Delta_c + \hbar \Delta_1 + \hbar^2 \Delta_2 + \dots$ satisfies $(\Delta^\infty)^2 = 0$. The associated quantum BRST differential is

$$s^q = (\Gamma^\infty, \cdot) + (\Delta^\infty)^L. \quad (6.82)$$

In the limit \hbar going to zero, it coincides with the classical differential \bar{s} .

In the extended antifield formalism, the anomalous Zinn-Justin equation can thus be written as a functional differential equation for the renormalized effective action. The derivations $\Delta_1, \Delta_2, \dots$ are guaranteed to exist due to the quantum action principles. They satisfy a priori cohomological restrictions due to the fact that the differential Δ^∞ is a formal deformation with deformation parameter \hbar of the differential Δ_c . In the context of chiral Yang-Mills theories, where $\Delta_c = 0$, an anomalous master equation of the form (6.81) for the renormalized effective action has appeared for the first time in [55].

6.2.3 Renormalization of local BRST cohomological classes

Associated quantum extension of a representative $\lambda_0 S$ of a classical BRST cohomological classes of the extended formalism satisfying $\bar{s}\lambda_0 S = 0$, or equivalently $[\Delta_c, \lambda_0] = 0$, are given by $\lambda_0 \Gamma^\infty$. By acting with λ_0 on (6.81), one gets

$$s^q \lambda_0 \Gamma^\infty = -[\Delta^\infty, \lambda_0] \Gamma^\infty. \quad (6.83)$$

The derivation $[\Delta^\infty, \lambda_0] = \sum_{n \geq L} \hbar^n \mu_n$ on the right hand side is of order at least \hbar , $L \geq 1$, because $[\Delta_c, \lambda_0] = 0$ and corresponds to the anomaly in the invariant renormalization of $\lambda_0 S$. The question then is whether there exists modified quantum extension $\lambda^\infty \Gamma^\infty$, with $\lambda^\infty = \lambda_0 + \hbar \lambda_1 + \dots$, such that

$$s^q \lambda^\infty \Gamma^\infty = 0 \iff [\Delta^\infty, \lambda^\infty] = 0. \quad (6.84)$$

This is for instance the case if λ_0 corresponds to the trivial cohomological class, $\lambda_0 = [\Delta_c, \nu_0]$. The searched for extension can then simply be taken to be $\lambda^\infty = [\Delta^\infty, \nu_0]$.

Because $[\Delta^\infty, [\Delta^\infty, \lambda_0]] = 0$, the lowest order part of the anomaly, μ_L is an s_{Δ_c} cocycle, $[\Delta_c, \mu_L] = 0$. Suppose that μ_L is a trivial solution to this equation, $\mu_L = -[\Delta_c, \lambda_L]$. The modified quantum extension $(\lambda_0 + \hbar^L \lambda_L) \Gamma^\infty$ allows to push the anomaly to order $L + 1$.

Hence, to lowest order in \hbar , the non trivial part of the anomaly in the renormalization of a classical cohomological class $H^g(s_{\Delta_c})$ is constrained to belong to $H^{g+1}(s_{\Delta_c})$. All the quantum information on this anomaly is encoded in the derivations $\Delta_1, \Delta_2, \dots$ and the whole discussion has been shifted from local functionals to derivations of functions of the coupling constants.

6.2.4 Quantum BRST cohomologies

The following lemma turns out to be extremely useful in the sequel. It concerns the insertion of BRST exact local functionals.

Lemma 2 *The insertion of a BRST exact local functional $\bar{s}\Xi = (S_{\text{gf}}, \Xi) + \Delta_c^L \Xi$ is equal to $s^q = (\Gamma^\infty, \cdot) + (\Delta^\infty)^L$ applied to a quantum extension of this local functional, up to a local insertion of higher order in \hbar ,*

$$[\bar{s}\Xi] \circ \Gamma^\infty = s^q \Xi^Q \circ \Gamma^\infty + \hbar I \circ \Gamma^\infty, \quad (6.85)$$

where $\Xi^Q = \Xi + O(\hbar)$ and I are local functionals.

Proof. If S^∞ is the sum of S_{gf} and the BRST finite breaking local counterterms needed to achieve (6.81), the action $S_\rho = S^\infty + \Xi \rho$ satisfies $\frac{1}{2}(S_\rho, S_\rho) + \Delta^\infty S_\rho = \bar{s}\Xi \rho + O(\hbar) + O(\rho^2)$. Applying the quantum action principle, we get $\frac{1}{2}(\Gamma_\rho, \Gamma_\rho) + \Delta^\infty \Gamma_\rho = \mathcal{D}(\rho) \circ \Gamma_\rho$. Putting ρ to zero, it follows from (6.81) that the local functional $\mathcal{D}(0) = 0$, so that $\mathcal{D}(\rho) = \mathcal{D}'(\rho) \rho$. Differentiation of the previous equation with respect to ρ and putting ρ to zero then implies $s^q \Xi^Q \circ \Gamma^\infty = \mathcal{D}'(0) \circ \Gamma^\infty$, for some local functional $\Xi^Q = \Xi + O(\hbar)$. At tree level, this equation implies that $\Delta'(0) = \bar{s}\Xi + O(\hbar)$, which gives the result. \square

Let us now establish the quantum analog of the classical equation (5.23), i.e.,

$$\left\{ \begin{array}{l} s^q B \circ \Gamma^\infty + \lambda^\infty \Gamma^\infty = 0, \\ [\Delta^\infty, \lambda^\infty] = 0, \end{array} \right\} \iff \left\{ \begin{array}{l} B \circ \Gamma^\infty = s^q C \circ \Gamma^\infty + \mu^\infty \Gamma^\infty, \\ \lambda^\infty = [\Delta^\infty, \mu^\infty]. \end{array} \right. \quad (6.86)$$

Proof. If the two equations on the right of the equivalence sign hold, the equations on the left just follow by applying s^q to the first equation on the right. Conversely,

suppose that the equations on the left hold. From the first equation at order zero in \hbar , we deduce that $\bar{s}B_0 + \lambda_0 S = 0$ with $[\Delta_c, \lambda_0] = 0$. According to (5.23), this implies $B_0 = \bar{s}C_0 + \mu_0 S$, with $\lambda_0 = [\Delta_c, \mu_0]$. Applying the quantum action principles and lemma 2, we get $B \circ \Gamma^\infty = s^q C_0^Q \circ \Gamma^\infty + \mu_0 \circ \Gamma^\infty + \hbar B' \circ \Gamma^\infty$. Injecting into the first equation gives $\hbar s^q B' \circ \Gamma^\infty + (\lambda^\infty - [\Delta^\infty, \mu_0])\Gamma^\infty = 0$. Because $\lambda^\infty - [\Delta^\infty, \mu_0] = \hbar \lambda'^\infty$, with $[\Delta^\infty, \lambda'^\infty] = 0$, we can factorize \hbar and iterate the reasoning, which proves (6.86). \square

If we put furthermore λ^∞ to zero in (6.86), we see that the general solution to the s^q cocycle condition is the insertion $\mu^\infty \Gamma^\infty$ with $[\Delta^\infty, \mu^\infty] = 0$, up to an s^q coboundary. We have thus proved the following theorem.

Theorem 10 *The cohomology of $s^q = (\Gamma^\infty, \cdot) + (\Delta^\infty)^L$ in the space of local insertions is isomorphic to the cohomology of $[\Delta^\infty, \cdot]$ in the space of graded right derivations in the couplings ξ^A that are formal power series in \hbar , the isomorphism being given by $[\lambda^\infty \Gamma^\infty] \longleftrightarrow [\lambda^\infty]$.*

These cohomology groups are defined to be the quantum BRST cohomology groups of the extended antifield formalism. They can be considered to be well defined versions of the formal quantum BRST cohomology of the standard Batalin-Vilkovisky formalism, involving the ill-defined second order operator $\Delta = (-)^{a+1} \frac{\delta^R}{\delta \phi^a(x)} \frac{\delta^R}{\delta \phi_a^*(x)}$ obtained through formal path integral manipulations.

7 Gauge parameter dependence

The problem of the gauge dependence of the effective action and of the renormalization group functions has been extensively studied in the mid seventies in the context of Yang-Mills theories [98, 99, 100, 16, 101]. An algebraic approach to the problem, independent of the renormalization scheme, has been proposed in [102]. On the assumption of the existence of an invariant renormalization scheme, extensions to generic, not necessarily power counting renormalizable theories have been considered in [34, 35, 36, 37] and more recently in [40, 41].

In this section, we combine the ideas of the above cited works and reinvestigate the problem in the general setting of the extended antifield formalism.

We assume that the gauge fixing fermion Ψ does not depend on any essential coupling, whereas, as stated before, that the minimal solution of the master equation only depends on the essential couplings. The gauge fixed action S_{gf} to be used as a starting point for perturbative quantization depends on the gauge parameters α^i , which we assume for simplicity to be bosonic, according to (3.6). This means that we have also to allow for a dependence of the effective action, the local insertions and the right derivations discussed so far on these gauge parameters.

7.1 Gauge parameter dependence of effective action and anomalies

According to the quantum action principle, $\partial_{\alpha^i}^R \Gamma^\infty = K_i \circ \Gamma_{\text{gf}}$, where $K_i = -(S_{\text{gf}}, \partial_{\alpha^i} \Psi)_{\phi_c, \tilde{\phi}^*} + O(\hbar) = -\bar{s} \partial_{\alpha^i}^R \Psi + O(\hbar)$. It follows from lemma 2 that this implies in a first step

$$\partial_{\alpha^i}^R \Gamma^\infty = s^q [-\partial_{\alpha^i}^R \Psi]^Q \circ \Gamma^\infty + \hbar K'_i \circ \Gamma^\infty. \quad (7.1)$$

Applying s^q and using (6.81), we get to lowest order in \hbar the consistency condition $[\Delta_1, \partial_{\alpha^i}^R] S_{\text{gf}} + \bar{s} K'_{i0} = 0$. Using $[\Delta_c, [\Delta_1, \partial_{\alpha^i}^R]] = 0$, equation (5.23) implies $K'_{i0} = -\rho_{i1} S_{\text{gf}} + \bar{s} N_{i1}$, with $[\Delta_1, \partial_{\alpha^i}^R] + [\Delta_c, \rho_{i1}] = 0$. The quantum action principle under the form $[\rho_{i1} S_{\text{gf}}] \circ \Gamma^\infty = \rho_{i1} \Gamma^\infty + \hbar L_i \circ \Gamma^\infty$, for a local insertion $L_i \circ \Gamma^\infty$, together with lemma 2 give $(\partial_{\alpha^i}^R + \hbar \rho_{i1}) \Gamma^\infty = s^q ([-\partial_{\alpha^i} \Psi]^Q + \hbar N_{i1}^Q) \circ \Gamma^\infty + \hbar^2 K''_i \circ \Gamma^\infty$. Defining $D_i^1 = \partial_{\alpha^i}^R + \hbar \rho_{i1}$, we have $[\Delta^\infty, D_i^1] = \hbar^2 \lambda_i + O(\hbar^3)$, so that $[\Delta_c, \lambda_i] = 0$, and the reasoning can be pushed to higher orders, with the result:

$$D_i^\infty \Gamma^\infty = s^q (L_i \circ \Gamma^\infty), \quad (7.2)$$

$$D_i^\infty = \partial_{\alpha^i}^R + \sum_{n=1} \hbar^n \rho_{in}, \quad [\Delta^\infty, D_i^\infty] = 0. \quad (7.3)$$

where $L_i = -\partial_{\alpha^i}^R \Psi + O(\hbar)$ and the coefficients ρ_{in}^A of the right derivations $\frac{\partial^R}{\partial \xi^A} \rho_{in}^A$ depend on the essential couplings ξ^A and the gauge couplings α^i . Note that the relation (6.86) cannot be used in this case, because D_i^∞ does not belong to the space of graded right derivations in the essential couplings alone (with a possible gauge parameter dependence).

Let us now analyze the equations (7.2) and (7.3) for each i separately and drop the subscript i . Let us define the functions $\xi_\alpha^A = \xi_\alpha^A(\xi^B, \alpha)$ through system of differential equations

$$\frac{d}{d\alpha} \xi_\alpha^A = \rho^A(\xi_\alpha^A, \alpha), \quad (7.4)$$

where $\rho^A = \sum_{n=1} \hbar^n \rho_n^A$. If we introduce a subscript α for all quantities where ξ^A has been replaced by ξ_α^A , equation (7.2) becomes

$$\frac{d}{d\alpha} \Gamma_\alpha^\infty = s_\alpha^q(L_\alpha \circ \Gamma_\alpha^\infty), \quad (7.5)$$

where $s_\alpha^q = (\Gamma_\alpha^\infty, \cdot) + \Delta_\alpha^\infty$, with $\Delta_\alpha^\infty = \frac{\partial^R}{\partial \xi_\alpha^A} \Delta_\alpha^{\infty A} \equiv \frac{\partial^R}{\partial \xi^B} \frac{\partial^R \xi^B}{\partial \xi_\alpha^A} \Delta_\alpha^{\infty A}$. Equation (7.3) then becomes

$$\frac{d}{d\alpha} \Delta_\alpha^\infty = 0. \quad (7.6)$$

We have thus proved:

Theorem 11 *For each gauge parameter separately, there exists redefinitions of the essential couplings by gauge parameter dependent terms of higher order in \hbar such that the variation of the effective action with respect to the gauge parameter is given by a quantum BRST coboundary, while the anomaly operator in terms of the redefined couplings Δ_α^∞ is independent of the gauge parameter.*

7.2 Integrability condition

By adapting the extended BRST technique of [102] to the present context, one can show

Lemma 3 *There exist local functionals $K_{[ij]}$ such that*

$$[D_i^\infty, D_j^\infty] \Gamma^\infty + s^q[K_{[ij]} \circ \Gamma^\infty] = 0. \quad (7.7)$$

Proof. If we introduce parameters λ^i of opposite Grassman parity to the α^i and define $S^e = S^\infty + \partial_{\alpha^i}^R \Psi \lambda^i$. Using $\partial_{\alpha^i}^R S_{\text{gf}} = \bar{s}(-\partial_{\alpha^i}^R \Psi)$ and $\frac{\partial^R \Psi}{\partial \alpha^i \alpha^j} \lambda^i \lambda^j = 0$, it follows that

$$\frac{1}{2}(S^e, S^e) + \Delta^\infty S^e + D_i^\infty S^e \lambda^i = \frac{1}{2}(\partial_{\alpha^i}^R \Psi \lambda^i, \partial_{\alpha^j}^R \Psi \lambda^j) + O(\hbar), \quad (7.8)$$

where $O(\hbar)$ is a local functional of order at least \hbar . Applying the quantum action principle, it follows that $\frac{1}{2}(\Gamma^e, \Gamma^e) + \Delta^\infty \Gamma^e + D_i^\infty \Gamma^e \lambda^i = \frac{1}{2}(\partial_{\alpha^i}^R \Psi \lambda^i, \partial_{\alpha^b}^R \Psi \lambda^b) \circ \Gamma^e + \hbar A \circ \Gamma^e$. Putting λ^i to zero and using (6.81), it follows that $A = A_i \lambda^i$. Differentiation with respect to λ^i and putting λ^i to zero gives $s^q(\partial_{\alpha^i}^R \Psi \circ \Gamma^\infty) + D_i^\infty \Gamma^\infty = \hbar A_i(0) \circ \Gamma^\infty$. Using (7.2), we deduce that $A_i(0) \circ \Gamma^\infty = s^q(L_i' \circ \Gamma^\infty)$, where $\hbar L_i' \circ \Gamma^\infty = \partial_{\alpha^i}^R \Psi \circ \Gamma^\infty + L_i \circ \Gamma^\infty$. If we now add to S^e the counterterm $-\hbar L_{i0}' \lambda^i$, we can absorb the lowest order contribution $A_i(0)$ up to terms of second order in \hbar or of first order in \hbar and of second order in λ^i . For the new Γ^e , we end up with $\frac{1}{2}(\Gamma^e, \Gamma^e) + \Delta^\infty \Gamma^e + D_i^\infty \Gamma^e \lambda^i = [\frac{1}{2} B_{[ij]}(\lambda) \lambda^i \lambda^j + \hbar^2 A_i'(0) \lambda^i] \circ \Gamma^e$, where $B_{[ij]}(\lambda) = (\partial_{\alpha^i}^R \Psi \lambda^i, \partial_{\alpha^j}^R \Psi \lambda^j) + O(\hbar)$.

Differentiation with respect to λ^i and putting λ^i to zero now gives $s^q(K_i \circ \Gamma^\infty) = A'_i(0) \circ \Gamma^\infty$, which implies that the lowest order contribution to $A'_i(0)$ can be absorbed by adding suitable counterterm proportional to λ^i and of order \hbar^2 . Going on in the same way, one can achieve:

$$\frac{1}{2}(\Gamma^e, \Gamma^e) + \Delta^\infty \Gamma^e + D_i^\infty \Gamma^e \lambda^i = \frac{1}{2} K_{[ij]}(\lambda) \circ \Gamma^e \lambda^i \lambda^j, \quad (7.9)$$

where $K_{[ij]}(\lambda) = (\partial_{\alpha^i}^R \Psi \lambda^j, \partial_{\alpha^j}^R \Psi \lambda^i) + O(\hbar)$.

Acting with $D_k^\infty \lambda^k$ on this equation, and using the same equation again, together with (7.3), $((\Gamma^e, \Gamma^e), \Gamma^e) = 0$, and $\Delta^{\infty 2} = 0$, we find $(\Gamma^e, \cdot) + \Delta^{\infty L} [\frac{1}{2} K_{[ij]}(\lambda) \lambda^i \lambda^j \circ \Gamma^e] + \frac{1}{2} [D_i, D_j] \Gamma^e \lambda^i \lambda^j = \frac{1}{2} D_k [K_{[ij]}(\lambda) \circ \Gamma^e] \lambda^i \lambda^j \lambda^k$. Differentiating with respect to λ^i and λ^j and putting λ to zero gives (7.7). \square

Note that $\partial_{\alpha^i}^R S_{\text{gf}} \lambda^i = -(S_{\text{gf}}, \partial_{\alpha^i}^R \Psi \lambda^i)$, and $\Delta_c \Psi = 0$ imply in particular that $\bar{s}(\partial_{\alpha^i}^R \Psi \lambda^i, \partial_{\alpha^j}^R \Psi \lambda^j) = 0$, so that both terms of (7.7) start indeed at order \hbar .

Because $[\partial_{\alpha^i}^R, \partial_{\alpha^j}^R] = 0$, $[D_i^\infty, D_j^\infty]$ belongs to the space of graded derivations in the essential couplings alone (with a possible gauge parameter dependence). Since furthermore, $[\Delta^\infty, [D_i^\infty, D_j^\infty]] = 0$ due to the graded Jacobi identity, (6.86) can now be applied with the result that

$$[D_i^\infty, D_j^\infty] = [\Delta^\infty, \rho_{ij}^\infty]. \quad (7.10)$$

In the case of the standard antifield formalism, supposed to be stable and anomaly free, so that in particular $\Delta^\infty = 0$, equation (7.10), is the integrability condition that states that the various redefinitions of the essential couplings can be made simultaneously [6]. The same holds in the general case, if one can prove that by suitable redefinitions of the D_i^∞ , the coefficients ρ_{ij}^∞ can be made to vanish. In this case, the dependence of the effective action on the all gauge parameters simultaneously is a quantum BRST coboundary, while the redefined anomaly operator is independent of all the gauge parameters.

In the anomaly free stable case, a quantum BRST coboundary, $(\Gamma^\infty, I \circ \Gamma^\infty)$ reduces to zero upon projection onto the physical states, so that the gauge parameter dependence of the effective action expressed in terms of the essential couplings on the quantum level is also trivial in this physical sense and not only in the cohomological sense. It would be of interest to analyze (i) in what sense a quantum BRST coboundary $s^q I \circ G^\infty = (\Gamma^\infty, I \circ G^\infty) + \Delta^\infty I \circ \Gamma^\infty$ is physically trivial and (ii) what can be concluded from (7.10) in the general case.

8 Renormalization group and Callan-Symanzik equations

8.1 Renormalization group equation

In addition to the gauge parameters, the effective action, the local insertions and the right derivations also contain a dependence on the renormalization scale μ .

Because $\mu\partial_\mu S_{\text{gf}} = 0$, the same derivation that has allowed to prove (7.2) now gives

$$D^\infty \Gamma^\infty = \hbar s^q(C \circ \Gamma^\infty), \quad (8.1)$$

$$D^\infty = \mu\partial_\mu + \sum_{n=1} \hbar^n \beta_n, \quad [\Delta^\infty, D^\infty] = 0. \quad (8.2)$$

The derivations $\beta_n = \frac{\partial^R}{\partial \xi^A} \beta_n^A$ only involve derivatives with respect to the essential couplings. If in another version of the renormalization group equation, there appear terms of the form $\frac{\partial^R \Gamma^\infty}{\partial \alpha^i} \rho_n^i$, they can always be absorbed by using (7.2) at the expense of a modification of the beta functions (of at least second order in \hbar) and of the quantum BRST exact term $s^q(C \circ \Gamma^\infty)$. This was first noted in the context of Yang-Mills theory in [98]. The same holds for any other redundant coupling. We will then make the following definition:

The physical beta functions β_n^A of the renormalization group equation are the coefficients of the derivatives $\frac{\partial^R}{\partial \xi^A}$ with respect to the essential couplings ξ^A in the renormalization group equation where the derivatives with respect to the redundant couplings have been eliminated.

Note that equation (8.2) can be written as $\mu\partial_\mu \Delta^\infty = \mu\partial_\mu (\sum_{n=1} \hbar^n \Delta_n) = -[\Delta^\infty, \sum_{n=1} \hbar^n \beta_n]$. It fixes the dependence of the anomaly coefficients $(\sum_{n=1} \hbar^{n-1} \Delta_n)$ on the renormalization point μ to be a quantum BRST coboundary, with the boundary term determined by the beta functions $\beta^A = \sum_{n=1} \hbar^n \beta_n^A$.

After integration of the renormalization group equations, i.e., after the determination of functions $\xi_\mu^A = \xi_\mu^A(\xi^B, \mu)$ defined through the differential equation

$$\mu \frac{d}{d\mu} \xi_\mu^A = \beta^A(\xi_\mu^A, \mu), \quad (8.3)$$

equations (8.1) and (8.2) in terms of the running couplings ξ_μ^A become

$$\mu \frac{d}{d\mu} \Gamma_\mu^\infty = \hbar s_\mu^q(C_\mu \circ \Gamma_\mu^\infty), \quad (8.4)$$

$$\mu \frac{d}{d\mu} \Delta_\mu^\infty = 0. \quad (8.5)$$

We have thus shown:

Theorem 12 *If the theory is expressed in terms of the running (essential) couplings ξ_μ^A , the variation of the effective action with respect to the renormalization scale is a quantum BRST coboundary, while the redefined anomaly operator Δ_μ^∞ does not depend on the renormalization scale.*

8.2 Gauge parameter dependence of renormalization group beta functions

If one follows [98] and commutes the functional operators of equations (7.2) and (8.1), one gets

$$[D^\infty, D_i^\infty]\Gamma^\infty = s^q E_i \quad (8.6)$$

where $E_i = D^\infty[L_i \circ \Gamma^\infty] - D_i^\infty[\hbar C \circ \Gamma^\infty] + (\hbar C \circ \Gamma^\infty, L_i \circ \Gamma^\infty)$.

Consider now the parameters $\alpha^{\bar{i}} = (\alpha^i, \mu)$, the constants ghosts $\lambda^{\bar{i}} = (\lambda^i, \Lambda)$ and the differentials $D_{\bar{i}}^\infty = (D_i, D)$. It then follows that (7.8) holds for the same S^e but with $D_i^\infty \lambda^i$ replaced by $D_{\bar{i}}^\infty \lambda^{\bar{i}}$. The proof that $[D_{\bar{i}}^\infty, D_{\bar{j}}^\infty] = [\Delta^\infty, \sigma_{\bar{i}\bar{j}}^\infty]$ then proceeds exactly as the proof of lemma 3 before and includes in particular the result

$$[D^\infty, D_i^\infty] = [\Delta^\infty, \sigma_i^\infty]. \quad (8.7)$$

Again, in the anomaly free standard formalism, one can deduce [6] from this equation (i) that the physical beta functions are gauge parameter independent if the redefined couplings ξ_α^A have been used, (ii) that the ρ^A are renormalization scale independent if the running couplings ξ_μ^A have been used, and (iii) that the absorption of the gauge parameter dependence and of the renormalization scale can be made simultaneously.

8.3 Power counting in the antifield formalism

In order to get the analogous results for the Callan-Symanzik equation, we have to replace the operator $\mu\partial_\mu$ by the generator of (broken) dilatation invariance. In the antifield formalism, power counting can be implemented canonically through the operator

$$S_\eta = \int d^n x L_\eta = \int d^n x \phi_a^* (d^{(a)} + x^\mu \partial_\mu) \phi^a, \quad (8.8)$$

where ϕ^a is a collective notation for the original fields and the local ghosts associated to the gauge symmetries, while $d^{(a)}$ is the canonical dimension of ϕ^a in units of inverse length. The bracket around the index a means that there is no additional summation. We have $(\phi^a(x), S_\eta) = (d^{(a)} + x^\mu \partial_\mu) \phi^a(x)$ and $(\phi_a^*(x), S_\eta) = (n - d^{(a)} + x^\mu \partial_\mu) \phi_a^*(x)$, so that the canonical dimension of the antifields is chosen to be $n - d^{(A)}$. It is then straightforward to verify that for any monomial $M(x)$ in the fields, the antifields and their derivatives of homogeneous dimension d^M ,

$$(M(x), S_\eta) = (d^M + x^\mu \partial_\mu) M(x). \quad (8.9)$$

8.4 Callan-Symanzik equation

8.4.1 No dimensionful coupling constants

For simplicity, we assume in a first stage that the coupling constants ξ^A as well as the redundant coupling constants all have dimension 0.

Because there are non dimensionful parameters, all the terms of the Lagrangian L of the (gauge fixed) solution of the extended master equation have dimension n . Hence,

$$(L, S_\eta) = (n + x^\mu \partial_\mu)L = \partial_\mu(x^\mu L). \quad (8.10)$$

Upon integration, we get

$$(S, S_\eta) = 0. \quad (8.11)$$

Furthermore, $\Delta_c S_\eta = 0 = \Delta^\infty S_\eta$ because S_η does not depend on ξ^A , which means $\bar{s}S_\eta = 0$. We have $(S^\infty, S_\eta) = O(\hbar) = (S^\infty, S_\eta) + \Delta^{\infty L} S_\eta$, so that the quantum action principle gives

$$(\Gamma^\infty, S_\eta) = \hbar \mathcal{B} \circ \Gamma^\infty = s^q S_\eta, \quad (8.12)$$

with \mathcal{B} a local functional of ghost number 0.

Applying s^q , we get the consistency condition $s^q(\mathcal{B} \circ \Gamma^\infty) = 0$, which implies, to lowest order in \hbar , $\bar{s}\mathcal{B}_0 = 0$ and hence

$$\mathcal{B}_0 = -\beta_1 S - \bar{s}\Xi_1, \quad [\Delta_c, \beta_1] = 0, \quad (8.13)$$

with $\beta_1 = \frac{\partial^R}{\partial \xi^A} \beta_1^A$. According to the quantum action principle, we can replace $\beta_1 S \circ \Gamma^\infty$ by $\beta_1 \Gamma^\infty$ and the difference will be the insertion of a local functional of order \hbar . Applying lemma 2, the local insertion $[\bar{s}\Xi_1] \circ \Gamma^\infty$ can be replaced by $s^q(\Xi_1^Q \circ \Gamma^\infty)$, and the difference will again be the insertion of a local functional of order \hbar . We thus get

$$s^q[S_\eta + \hbar \Xi_1^Q \circ \Gamma^\infty] + \hbar \beta_1 \Gamma^\infty = \hbar^2 \mathcal{B}' \circ \Gamma^\infty. \quad (8.14)$$

Acting with s^q on (8.14), using $[\Delta_c, \beta_1] = 0$, we get the consistency condition $\hbar[\Delta^\infty \beta_1] \Gamma^\infty + \hbar^2 s^q(\mathcal{B}' \circ \Gamma^\infty) = 0$, giving to lowest order $[\Delta_1, \beta_1] S + \bar{s}\mathcal{B}'_0 = 0$. As in the previous section, this means that the reasoning can be pushed to all orders:

$$s^q[S_\eta - \hbar \Xi^\infty \circ \Gamma^\infty] + \hbar \beta^\infty \Gamma^\infty = 0, \quad (8.15)$$

$$[\Delta^\infty, \beta^\infty] = 0. \quad (8.16)$$

with $\Xi^\infty = \sum_{n=1} \hbar^{n-1} \Xi_n^Q$ and $\beta^\infty = \sum_{n=1} \hbar^{n-1} \beta_n$. Note that if $\beta^\infty = [\Delta^\infty, \sigma^\infty]$, the term $\beta^\infty \Gamma^\infty$ in (8.15) can be absorbed by the redefinition $\Xi^\infty \circ \Gamma^\infty \longrightarrow \Xi^\infty \circ \Gamma^\infty - \sigma^\infty \Gamma^\infty$. We then get from (8.16):

Theorem 13 *The right derivation built out of the beta functions of the Callan-Symanzik equation defines a non trivial quantum BRST cocycle in ghost number 0.*

8.4.2 Digression

Let us consider for a moment the following particular case.

(i) All the antibracket maps encoded in f^A are zero so that $\Delta_c = 0$, $\bar{s} = s = (S(\xi), \cdot)$. This happens for instance if one couples only the BRST cohomological classes in ghost number 0 and if the Kluberg-Stern and Zuber conjecture [16], stating that these cohomological classes can be described independently of the antifields, is valid. This guarantees stability of the standard antifield formalism.

(ii) The theory is anomaly free, $\frac{1}{2}(\Gamma^\infty, \Gamma^\infty) = 0$.

(iii) The only possibility (for instance for power counting reasons) for Ξ_n is $\Xi_n = -\gamma_n \int d^n x \phi_a^* \phi^a$, so that Ξ_n is linear in the quantum fields and $[s\Xi_n] \circ \Gamma^\infty$ can be replaced, according to the quantum action principle, at each stage in \hbar by (Γ^∞, Ξ_n) up to the insertion of a local polynomial of higher order in \hbar .

Equation (8.15) then reduces to

$$(\Gamma^\infty, S_\eta^\infty) + \hbar \frac{\partial^R \Gamma^\infty}{\partial \xi^A} \beta^A = 0, \quad (8.17)$$

with $S_\eta^\infty = \int d^n x \phi_a^* (d^{(a)} + \hbar \gamma + x \cdot \partial) \phi^a$, or explicitly

$$\begin{aligned} & \int d^n x \left[\frac{\delta^R \Gamma^\infty}{\delta \phi^a(x)} (d^{(a)} + \hbar \gamma + x \cdot \partial) \phi^a(x) \right. \\ & \left. + \frac{\delta^R \Gamma^\infty}{\delta \phi_a^*(x)} (n - d^{(a)} - \hbar \gamma + x \cdot \partial) \phi_a^*(x) \right] + \hbar \frac{\partial^R \Gamma^\infty}{\partial \xi^A} \beta^A = 0. \end{aligned} \quad (8.18)$$

After putting to zero the antifields, the second part of the first integral vanishes and this equation is a familiar form of the Callan-Symanzik equation (see eg. [103]) in the massless case, with anomalous dimension $\gamma = \sum_{k=1} \hbar^{k-1} \gamma_k$ for the fields.

8.4.3 Remarks on explicit x dependence.

Note that S_η is the generator of the dilatation symmetry of the theory. If it corresponds to a non trivial element of $H^{-1,n}(s|d)$, the question arises whether it should be coupled with a constant ghost in the extended solution $S(\xi)$ as in [65, 104]. This depends on whether or not we allow for explicit x dependence in the local functionals and the cohomology classes of s we are initially computing and then coupling to the solution of the master equation.

In the previous section, we have supposed that there is no explicit x dependence in these functionals and cohomology classes, because if we assume the absence of dimensionful couplings, we cannot control translation invariance through a corresponding cohomology class, its generator $S_\mu = \int d^n x \phi^* \partial_\mu \phi$ being of dimension 1.

We will assume here that one can apply the quantum action principles in the case of an explicit x dependence of the variation as in (8.12), at the price of allowing a priori for an explicit x dependence of the inserted local functional \mathcal{B} . This assumption needs to be checked by a more careful analysis of the renormalization properties of the model which is beyond the scope of this review.

In order to prove then that \mathcal{B} in eq (8.12) does not depend explicitly on x , we use translation invariance: classical translation invariance is expressed through $(S, S_\mu) = 0$ with quantum version $(\Gamma^\infty, S_\mu) = \hbar \mathcal{D}_\mu \circ \Gamma^\infty$, where the dimension of \mathcal{D}_μ is 1, because there are no dimensionful parameters in the theory. Applying (\cdot, S_μ) to (8.12), using the graded Jacobi identity for the antibracket, the commutation relation $(S_\eta, S_\mu) = -S_\mu$ and the result on the dimension of \mathcal{D}_μ , i.e., the relation $(\mathcal{D}_\mu, S_\eta) = \mathcal{D}_\mu$, one finds to lowest order $(\mathcal{B}, S_\mu) = 0$. This means that $(\partial_\mu - \partial/\partial x^\mu)\mathcal{B} = 0$, and since $\partial_\mu \mathcal{B} = 0$, it shows that \mathcal{B} does not depend explicitly on x .

In the general case where we allow for dimensionful couplings considered below, we will assume that the theory is translation invariant and that the generator S_μ is coupled through the constant translation ghosts ξ^μ . One can then show that the local cohomology of the BRST operator in form degree n for the extended theory can be chosen to be independent of both x^μ and ξ^μ [105]. In the same way, Lorentz invariance can then be controlled inside the formalism by coupling the appropriate generator.

8.4.4 General broken case

We will now allow for coupling constants ξ^A, α^i of all possible dimensions $d^{(A)}, d^{(i)}$ in the theory, which could be negative in the case of effective field theories. We have

$$(L, S_\eta) + \frac{\partial^R L}{\partial \xi^A} d^{(A)} \xi^A + \frac{\partial^R L}{\partial \alpha^i} d^{(i)} \alpha^i = \partial_\mu (x^\mu L). \quad (8.19)$$

Integrating, one gets

$$\mathcal{C}S = 0, \quad (8.20)$$

with $\mathcal{C} = (\cdot, S_\eta) + \frac{\partial^R \cdot}{\partial \xi^A} d^{(A)} \xi^A + \frac{\partial^R \cdot}{\partial \alpha^i} d^{(i)} \alpha^i$.

Using (3.6) and $\Delta_c S_\eta = 0 = \Delta_c \partial_{\alpha^i}^R \Psi$, (8.20) becomes

$$\bar{s}(S_\eta - \partial_{\alpha^i}^R \Psi d^{(i)} \alpha^i) + \frac{\partial^R S}{\partial \xi^A} d^{(A)} \xi^A = 0. \quad (8.21)$$

Applying \bar{s} , we find $[\Delta_c, \beta_0] = 0$, with $\beta_0 = \frac{\partial^R S}{\partial \xi^A} d^{(A)} \xi^A$. The quantum version of this equation is

$$s^q[S_\eta - (\partial_{\alpha^i}^R \Psi)^Q d^{(i)} \alpha^i \circ \Gamma^\infty] + \beta_0 \Gamma^\infty = \hbar \mathcal{B} \circ \Gamma^\infty. \quad (8.22)$$

Applying now s^q , the consistency condition to lowest order implies $\bar{s}\mathcal{B}_0 = 0$. We can then get as in the previous section the general form of the integrated Callan-Symanzik equation:

$$s^q[S_\eta - \Xi \circ \Gamma^\infty] + \beta^\infty \Gamma^\infty = 0, \quad (8.23)$$

$$[\Delta^\infty, \beta^\infty] = 0, \quad (8.24)$$

with $\Xi = -\partial_{\alpha^i}^R \Psi d^{(i)} \alpha^i + O(\hbar)$ and $\beta^\infty = \Sigma_{n=0} \hbar^n \beta_n$. Hence, by defining $d^{(A)} \xi^A$ to be the tree level contribution of the beta function, theorem 13 extends to the general case.

9 Refined anomaly consistency condition

We have seen in section 6 that the non trivial breakings of the extended master equation are constrained by the cohomology of \bar{s} in ghost number 1 in the space of local functionals. In the same way, one can study the constraints on the anomalies to an invariant renormalization of symmetric integrated and non integrated operators [106]. In the cohomological reformulation, symmetric integrated or non integrated operators correspond to BRST cohomological classes in ghost number 0 in the space of local functionals, respectively in the space of local functions. More generally, one can be interested in the anomalies appearing during the renormalization of a BRST cohomological class in ghost number g in these spaces. The non trivial anomalies that can appear can be shown to belong to the corresponding BRST cohomological classes in ghost number $g + 1$. Some aspects of the renormalization of integrated BRST cohomological classes in the extended antifield formalism have already been discussed in section 6.2.3.

The computation in the space of local functionals of the cohomology of the standard BRST differential s with antifields can be reduced to the computation of a relative cohomological group in the space of local n -forms by introducing the space-time exterior derivative d . It is then related to the cohomology of s in the space of form valued local functions through descent equations [107, 108, 109, 110]. The same holds for the BRST differential \bar{s} associated to the extended antifield formalism.

More generally, we will call the relative cohomological groups $H^{g,p}(s|d)$ and $H^{g,p}(\bar{s}|d)$ in ghost number g and form degree p local BRST cohomological groups. A more detailed analysis of the descent equations [111] shows that these groups are characterized by two integers, the length d of their descents and the length l of their lifts.

9.1 Characterization of local BRST cohomological classes

In this and the following section, we decompose the space Ω of Lorentz-invariant polynomials or formal powers series in the dx^μ , the couplings ξ^A , the fields, antifields and their derivatives into the direct sum of the constants and the remaining part, $\Omega = \mathbf{R} \oplus \Omega_+$. We have $\bar{s}\alpha = 0 = d\alpha$, for a constant α and $d\Omega_+ \subset \Omega_+$. We furthermore assume that if $\bar{s}\omega = \alpha$ for a constant α , then $\alpha = 0$, which amounts to assuming that the equations of motions are consistent (see the discussion in chapter 9 of [33]). This means that $\bar{s}\Omega_+ \subset \Omega_+$. Hence, we can consider the cohomological groups $H(\bar{s}, \Omega_+)$, $H(\bar{s}|d, \Omega_+)$ and $H(d, \Omega_+)$.

By analyzing the cohomological groups $H(\bar{s}|d)$ (the space Ω_+ being always understood in the following) using descent equations, one can prove [111] that the elements of these groups can be classified into chains of length r with an obstruction to further lifts and chains of length s whose lifts are unobstructed, i.e., chains with a non trivial element in degree n (see also [112, 113, 114]; we follow here the notations of the review [33], where explicit proofs of the statements below can be found). More

precisely, we have $H^p(d) = 0$, $p \leq n-1$, and there exists a basis

$$\{[h_{i_r}^0], [\hat{h}_{i_r}], [e_{\alpha_s}^0]\} \quad (9.1)$$

of $H(\bar{s})$, for $r = 0, \dots, n-1$, $s = 0, \dots, n$, such that a corresponding basis of $H(\bar{s}|d)$ is given by

$$\{[h_{i_r}^q], [e_{\alpha_s}^p]\} \quad (9.2)$$

for $q = 0, \dots, r$ and $p = 0, \dots, s$, with

$$\begin{aligned} \bar{s}h_{i_r}^{r+1} + dh_{i_r}^r &= \hat{h}_{i_r}, \\ \bar{s}h_{i_r}^r + dh_{i_r}^{r-1} &= 0, \\ &\vdots \\ \bar{s}h_{i_r}^1 + dh_{i_r}^0 &= 0, \\ \bar{s}h_{i_r}^0 &= 0, \end{aligned} \quad (9.3)$$

and

$$\begin{aligned} \text{form degree } e_{\alpha_s}^s &= n, \quad de_{\alpha_s}^s = 0, \\ \bar{s}e_{\alpha_s}^s + de_{\alpha_s}^{s-1} &= 0, \\ &\vdots \\ \bar{s}e_{\alpha_s}^1 + de_{\alpha_s}^0 &= 0, \\ \bar{s}e_{\alpha_s}^0 &= 0. \end{aligned} \quad (9.4)$$

The cohomological group $H(\bar{s})$ can thus be decomposed into elements $[e_{\alpha_s}^0]$ that are bottoms of unobstructed chains of length s , elements $[h_{i_r}^0]$ that are bottoms of obstructed chains of length r and obstructions $[\hat{h}_{i_r}]$ to chains of length r .

For the cohomological group $H(\bar{s}|d)$, the element $[h_{i_r}^l]$ is said to be the element of level l of a chain of length r with obstruction; it has l non trivial descents and $r-l$ non trivial lifts ; while the element $[e_{\alpha_s}^l]$ is said to be the element of level l of a chain of length s without obstructions, it has l non trivial descents and $s-l$ non trivial lifts.

One can furthermore show that the general solution to a set of descent equations involving at most l steps, $\bar{s}\omega^l + d\omega^{l-1} = 0$, $\bar{s}\omega^{l-1} + d\omega^{l-2} = 0, \dots, \bar{s}\omega^0 = 0$, can be written in terms of the above elements as

$$\omega^l = \sum_{q=0}^l \sum_{r=l-q}^{n-1} \lambda_q^{i_r} h_{i_r}^{l-q} + \sum_{p=0}^l \sum_{s=l-p}^n \mu_p^{\alpha_s} e_{\alpha_s}^{l-p} + \bar{s}\eta^l + d[\eta^{l-1} + \sum_{r=0}^{n-1} \nu^{(l)i_r} h_{i_r}^r], \quad (9.5)$$

with $\eta^{-1} = 0$. This means that a \bar{s} modulo d cocycle which has l non trivial descents, is a linear combination of all elements of the chains (9.3) and (9.4) which have l or less non trivial descents. If such a linear combination is \bar{s} modulo d trivial, the coefficients of the linear combination must vanish, i.e.,

$$\sum_{q=0}^l \sum_{r=l-q}^{n-1} \lambda_q^{i_r} h_{i_r}^{l-q} + \sum_{p=0}^l \sum_{s=l-p}^n \mu_p^{\alpha_s} e_{\alpha_s}^{l-p} = \bar{s}(\) + d(\). \quad (9.6)$$

implies that $\lambda_q^{i_r} = 0 = \mu_p^{\alpha_s}$.

9.2 Lengths of descent and lifts of lowest order anomalies

Let us now investigate the anomalies in the BRST invariant renormalization of a chain (9.4) of length s without obstructions. We follow the approach of [115], which consists in considering simultaneously the anomalies for a whole chain of descent equations (for a review, see [47]). Using the quantum action principles, one can show the analog of lemma 2 for a local form a :

$$\bar{s}a \circ \Gamma^\infty = s^q a^Q \circ \Gamma^\infty + \hbar b \circ \Gamma^\infty, \quad (9.7)$$

for some local form b . When applied to the chain (9.4), we find that the quantum version of this chain is

$$\begin{aligned} \text{form degree } e_{\alpha_s}^s \circ \Gamma^\infty &= n, \quad de_{\alpha_s}^s \circ \Gamma^\infty = 0, \\ s^q e_{\alpha_s}^s \circ \Gamma^\infty + de_{\alpha_s}^{s-1} \circ \Gamma^\infty &= \hbar a_{\alpha_s}^s \circ \Gamma^\infty, \\ &\vdots \\ s^q e_{\alpha_s}^1 \circ \Gamma^\infty + de_{\alpha_s}^0 \circ \Gamma^\infty &= \hbar a_{\alpha_s}^1 \circ \Gamma^\infty, \\ s^q e_{\alpha_s}^0 \circ \Gamma^\infty &= \hbar a_{\alpha_s}^0 \circ \Gamma^\infty, \end{aligned} \quad (9.8)$$

where $a_{\alpha_s}^l \circ \Gamma^\infty = a_{\alpha_s}^s + O(\hbar)$ for a local function $a_{\alpha_s}^l$. Applying s^q , we get the consistency condition,

$$\begin{aligned} \text{form degree } a_{\alpha_s}^s \circ \Gamma^\infty &= n, \quad da_{\alpha_s}^s \circ \Gamma^\infty = 0 \\ s^q a_{\alpha_s}^s \circ \Gamma^\infty + da_{\alpha_s}^{s-1} \circ \Gamma^\infty &= 0, \\ &\vdots \\ s^q a_{\alpha_s}^1 \circ \Gamma^\infty + da_{\alpha_s}^0 \circ \Gamma^\infty &= 0, \\ s^q a_{\alpha_s}^0 \circ \Gamma^\infty &= 0. \end{aligned} \quad (9.9)$$

At lowest order in \hbar , we get

$$\begin{aligned} \text{form degree } a_{\alpha_s}^s &= n, \quad da_{\alpha_s}^s = 0 \\ \bar{s}a_{\alpha_s}^s + da_{\alpha_s}^{s-1} &= 0, \\ &\vdots \\ \bar{s}a_{\alpha_s}^1 + da_{\alpha_s}^0 &= 0, \\ \bar{s}a_{\alpha_s}^0 &= 0. \end{aligned} \quad (9.10)$$

Using equation (9.5) and the fact that the form degree of $a_{\alpha_s}^s$ is n , it follows that

$$a_{\alpha_s}^l = \sum_{p=0}^l \mu_p^{\beta_{s-p}} e_{\beta_{s-p}}^{l-p} + \bar{s}\eta_{\alpha_s}^l + d[\eta_{\alpha_s}^{l-1} + \sum_{r \geq 0} \nu_{\alpha_s}^{(l)i_r} h_{i_r}^r], \quad (9.11)$$

for $l = 0, \dots, s$. This gives our first result:

The anomaly in the renormalization of an element of level l of a chain of length s without obstructions involves at most elements of chains of the same type with less non trivial descents and the same number of non trivial lifts.

For the anomalies for a chain with obstruction (9.3), we get,

$$\begin{aligned}
s^q h_{i_r}^{r+1} \circ \Gamma^\infty + dh_{i_r}^r \circ \Gamma^\infty &= \hat{h}_{i_r} \circ \Gamma^\infty + \hbar a_{i_r}^{r+1} \circ \Gamma^\infty, \\
s^q h_{i_r}^r \circ \Gamma^\infty + dh_{i_r}^{r-1} \circ \Gamma^\infty &= \hbar a_{i_r}^r \circ \Gamma^\infty, \\
&\vdots \\
s^q h_{i_r}^1 \circ \Gamma^\infty + dh_{i_r}^0 \circ \Gamma^\infty &= \hbar a_{i_r}^1 \circ \Gamma^\infty, \\
s^q h_{i_r}^0 \circ \Gamma^\infty &= \hbar a_{i_r}^0 \circ \Gamma^\infty,
\end{aligned} \tag{9.12}$$

We also have

$$s^q \hat{h}_{i_r} \circ \Gamma^\infty = -\hbar \hat{a}_{i_r} \circ \Gamma^\infty. \tag{9.13}$$

Applying s^q gives $s^q \hat{a}_{i_r} \circ \Gamma^\infty = 0$ and then to lowest order, $\bar{s} \hat{a}_{i_r} = 0$. Applying now s^q to the chain (9.12) gives

$$\begin{aligned}
s^q a_{i_r}^{r+1} \circ \Gamma^\infty + da_{i_r}^r \circ \Gamma^\infty &= \hat{a}_{i_r} \circ \Gamma^\infty, \\
s^q a_{i_r}^r \circ \Gamma^\infty + da_{i_r}^{r-1} \circ \Gamma^\infty &= 0, \\
&\vdots \\
s^q a_{i_r}^1 \circ \Gamma^\infty + da_{i_r}^1 \circ \Gamma^\infty &= 0, \\
s^q a_{i_r}^0 \circ \Gamma^\infty &= 0,
\end{aligned} \tag{9.14}$$

and to lowest order,

$$\begin{aligned}
\bar{s} a_{i_r}^{r+1} + da_{i_r}^r &= \hat{a}_{i_r}, \\
\bar{s} a_{i_r}^r + da_{i_r}^{r-1} &= 0, \\
&\vdots \\
\bar{s} a_{i_r}^1 + da_{i_r}^1 &= 0, \\
\bar{s} a_{i_r}^0 &= 0,
\end{aligned} \tag{9.15}$$

On the one hand, it follows from (9.5) that

$$d_{i_r}^l = \sum_{q=0}^l \sum_{r'=r-q}^{n-1} \lambda_q^{j_{r'}} h_{j_{r'}}^{l-q} + \sum_{p=0}^l \sum_{s=r-p+1}^n \mu_p^{\beta_s} e_{\beta_s}^{l-p} + \bar{s} \eta_{i_r}^l + d[\eta_{i_r}^{l-1} + \sum_{r'=0}^{n-1} \nu_{i_r}^{(l)j_{r'}} h_{j_{r'}}^{r'}] \tag{9.16}$$

for $l = 0, \dots, r$, while on the other hand, the cohomology of \bar{s} implies

$$\hat{a}_{i_r} = \sum_{r'=0}^{n-1} \alpha_{i_r}^{j_{r'}} \hat{h}_{j_{r'}} + \sum_{r'=0}^{n-1} \beta_{i_r}^{j_{r'}} h_{i_r}^0 + \sum_{s=0}^n \gamma_{i_r}^{\alpha_s} e_{\alpha_s}^0 + \bar{s} \hat{o}_{i_r}. \tag{9.17}$$

Applying d to (9.16) at $l = r$ gives

$$\begin{aligned}
da_{i_r}^r &= - \sum_{q=0}^r \sum_{r'=r-q+1}^{n-1} \lambda_q^{i_r'} \bar{s} h_{i_r'}^{r-q+1} - \sum_{p=0}^r \sum_{s=r-p+1}^n \mu_p^{\beta_s} \bar{s} e_{\beta_s}^{r-p+1} - \bar{s} d \eta_{i_r}^r \\
&\quad + \sum_{q=0}^r \lambda_q^{i_r-q} (-\bar{s} h_{i_r-q}^{r-q+1} + \hat{h}_{i_r-q}).
\end{aligned} \tag{9.18}$$

Injecting now (9.17) and (9.18) into the first equation of (9.15) gives first of all $\beta_{i_r}^{j_{r'}} = 0 = \gamma_{i_r}^{\alpha_s}$ and also

$$\hat{a}_{i_r} = \sum_{q=0}^r \lambda_q^{j_{r-q}} \hat{h}_{j_{r-q}} + \bar{s} \hat{o}_{i_r}, \quad (9.19)$$

and then,

$$\begin{aligned} \bar{s}(a_{i_r}^{r+1} - \sum_{q=0}^r \sum_{r'=r-q+1}^{n-1} \lambda_q^{i_{r'}} h_{i_{r'}}^{r-q+1} - \sum_{p=0}^r \sum_{s=r-p+1}^n \mu_p^{\beta_s} e_{\beta_s}^{r-p+1} \\ - \sum_{q=0}^r \lambda_q^{i_{r-q}} h_{i_{r-q}}^{r-q+1} - d\eta_{i_r}^r - \hat{o}_{i_r}) = 0, \end{aligned} \quad (9.20)$$

so that, using the cohomology of \bar{s} ,

$$\begin{aligned} a_{i_r}^{r+1} = \sum_{q=0}^{r+1} \sum_{r'=r-q}^{n-1} \lambda_q^{i_{r'}} h_{i_{r'}}^{r-q+1} + \sum_{p=0}^{r+1} \sum_{s=r-p+1}^n \mu_p^{\beta_s} e_{\beta_s}^{r-p+1} \\ + d[\eta_{i_r}^r + \sum_{r'=0}^{n-1} \nu_{i_r}^{(r+1)j_{r'}} h_{j_{r'}}^{r'}] + \hat{o}_{i_r} + \bar{s}\eta_{i_r}^{r+1}. \end{aligned} \quad (9.21)$$

Our second result is then:

The anomaly in the renormalization of an element of level l of a chain of length r with obstructions involves at most elements of chains with obstructions with less non trivial descents and more non trivial lifts and elements of chains without obstructions with less non trivial descents and strictly more non trivial lifts.

Let us now rewrite (9.11) and (9.16) at $l=0$ as

$$a_{\alpha_s}^0 = \mu_0^{\beta_s} e_{\beta_s}^0 + \bar{s}[\eta_{\alpha_s}^0 - \sum_{r=0}^{n-1} \nu_{\alpha_s}^{(0)i_r} h_{i_r}^{r+1}] + \sum_{r=0}^{n-1} \nu_{\alpha_s}^{(0)i_r} \hat{h}_{i_r}, \quad (9.22)$$

$$a_{i_r}^0 = \sum_{r'=r}^{n-1} \lambda_0^{j_{r'}} h_{j_{r'}}^0 + \sum_{s=r+1}^n \mu_0^{\beta_s} e_{\beta_s}^0 + \bar{s}[\eta_{i_r}^0 - \sum_{r'=0}^{n-1} \nu_{i_r}^{(0)j_{r'}} h_{j_{r'}}^{r'+1}] + \sum_{r'=0}^{n-1} \nu_{i_r}^{(0)j_{r'}} \hat{h}_{j_{r'}}. \quad (9.23)$$

Combined with (9.19), our third result on anomalies in the renormalization of elements of $H(\bar{s})$ is accordingly:

The anomaly in the renormalization of obstructions to chains of length r involves at most obstructions to shorter chains; the anomaly in the renormalization of bottoms of unobstructed chains of length s involves at most bottoms of unobstructed chains of the same length and obstructions to chains of all possible lengths; the anomaly in the renormalization of bottoms of obstructed chains of length r involves at most bottoms of obstructed chains of greater length, bottoms of unobstructed chains of strictly greater length and obstructions to chains of all possible lengths.

9.3 Differentials associated to one loop anomalies

A different, more compact, way to formulate and prove the results of section 9.1 and 9.2 is to use the exact couple describing the descent and the associated spectral sequence [111].

Indeed, the diagram

$$\begin{array}{ccc} H(\bar{s}|d) & \xrightarrow{\mathcal{D}} & H(\bar{s}|d) \\ i_0 \swarrow & & \searrow l_0 \\ & H(\bar{s}) & \end{array} \quad (9.24)$$

can be shown to be exact at all corners. The various maps are defined as follows: i_0 is the map which consists in regarding an element of $H(\bar{s})$ as an element of $H(\bar{s}|d)$, $i_0 : H(\bar{s}) \longrightarrow H(\bar{s}|d)$, with $i_0[a] = [a]$. It is well defined because every \bar{s} cocycle is a \bar{s} cocycle modulo d and every \bar{s} coboundary is a \bar{s} coboundary modulo d . The descent homomorphism $\mathcal{D} : H^{k,l}(\bar{s}|d) \longrightarrow H^{k+1,l-1}(\bar{s}|d)$ with $\mathcal{D}[a] = [b]$, if $\bar{s}a + db = 0$ is well defined because of the triviality of the cohomology of d in form degree $p \leq n-1$. Finally, the map $l_0 : H^{k+1,l-1}(\bar{s}|d) \longrightarrow H^{k+1,l}(\bar{s})$ is defined by $l_0[a] = [da]$. It is well defined because the relation $\{\bar{s}, d\} = 0$ implies that it maps cocycles to cocycles and coboundaries to coboundaries.

To such an exact couple $(H(\bar{s}|d), K_0 = H(\bar{s}))$, one can associate in a standard way derived exact couples $(\mathcal{D}^r H(\bar{s}|d), K_r)$,

$$\begin{array}{ccc} \mathcal{D}^r H(\bar{s}|d) & \xrightarrow{\mathcal{D}} & \mathcal{D}^r H(\bar{s}|d) \\ i_r \swarrow & & \searrow l_r \\ & K_r & \end{array} \quad (9.25)$$

and a spectral sequence $K_{r+1} = H(d_r, K_r)$, with $K_0 \equiv H(\bar{s})$. The maps of these exact couples are defined recursively as follows: the map $d_{r-1} = l_{r-1} \circ i_{r-1}$ can be shown to be a differential, the map i_r is the map induced by i_{r-1} in K_r , while $l_r \mathcal{D}[a] = l_{r-1}[a]$.

Explicitly, the differential $d_0 : K_0 \longrightarrow K_0$ is defined by $d_0[a] = [da]$, where $\bar{s}a = 0$. An element $k_r \in K_r$ is identified with the equivalence class $[a]_r$ of an element $[a] \in H(\bar{s})$, where $[a] \sim_r [a']$ if $[a] - [a'] \in \oplus_{q=0}^{r-1} \text{im } d_q$. The relations $d_q k_r = 0$, $q = 0, \dots, r-1$ mean that k_r is a bottom that can be lifted at least r times, i.e., there exist c_1, \dots, c_{r+1} such that $\bar{s}a = 0, da + \bar{s}c_1 = 0, \dots, dc_{r-1} + \bar{s}c_r = 0$. Then, the differential d_r is defined by $d_r k_r = [dc_r]_r$.

Because there are no forms of form degree higher than n , $\mathcal{D}^{n+1} H(\bar{s}|d) = 0$ and $d_n \equiv 0$ so that the construction stops at $r = n$.

The space of local forms Ω is decomposed as $\Omega = E_0 \oplus G \oplus \bar{s}G \oplus \mathbf{R}$, with $E_0 \simeq K_0 = H(\bar{s})$. If we define $E_r, F_{r-1} \subset E_{r-1}$ through $E_{r-1} = E_{r-1} \oplus E_r \oplus d_{r-1}F_{r-1}$ with $E_r \simeq K_r$, we get the decomposition

$$E_0 = F_0 \oplus \dots \oplus F_{n-1} \oplus E_n \oplus d_{r-1}F_{r-1} \oplus \dots \oplus d_0 F_0. \quad (9.26)$$

The $e_{\alpha_s}^0$ are elements of a basis of E_n that can be lifted s times before hitting form degree n , i.e., that are of form degree $n-s$, while \hat{h}_{i_r} and $h_{i_r}^0$ are elements of a basis of $d_r F_r$ and F_r respectively. This sums up the results of section 9.1.

Let us now define the linear map

$$\delta_0 : H^g(\bar{s}) \longrightarrow H^{g+1}(\bar{s}), \quad (9.27)$$

$$\delta_0[a] = [b], \text{ where } s^q a \circ \Gamma^\infty = \hbar b \circ \Gamma^\infty. \quad (9.28)$$

The map associates to a given BRST cohomological class the non trivial order \hbar , i.e., 1 loop contribution of its anomaly.

The map is well defined, because the consistency condition implies that $\bar{s}b = 0$. Furthermore, if $a = \bar{s}c$, $a \circ \Gamma^\infty = s^q c \circ \Gamma^\infty + \hbar d \circ \Gamma^\infty$, so that $s^q a \circ \Gamma^\infty = \hbar s^q d \circ \Gamma^\infty$, meaning that $b = \bar{s}d$. This implies that the map does not depend on the choice of the representative. In addition, this map is a differential

$$\delta_0^2 = 0. \quad (9.29)$$

Indeed, if $[a] = \delta_0[c]$, we have $a \circ \Gamma^\infty = \frac{1}{\hbar} s^q c \circ \Gamma^\infty$. It follows that $s^q a \circ \Gamma^\infty = 0$. A BRST cohomological class which is a δ_0 -cocycle has no 1-loop anomaly, while a BRST cohomological class which is a δ_0 -coboundary is the 1-loop anomaly of some other BRST cohomological class.

We thus have two differentials in $K_0 = H(\bar{s})$, d_0 introduced above and δ_0 . These differentials anticommute,

$$\{d_0, \delta_0\} = 0. \quad (9.30)$$

Indeed, if $s^q a \circ \Gamma^\infty = \hbar b \circ \Gamma^\infty$, $d_0 \delta_0[a] = d_0[b] = [db]$, while $\delta_0 d_0[a] = \delta_0[da]$, and $s^q(da \circ \Gamma^\infty) = s^q d(a \circ \Gamma^\infty + \hbar c \circ \Gamma^\infty) = -ds^q(a \circ \Gamma^\infty + \hbar c \circ \Gamma^\infty) = -\hbar d(b \circ \Gamma^\infty + s^q c \circ \Gamma^\infty)$, so that $\delta_0[da] = [-d(b + \bar{s}c)] = -[db]$.

The relation $d_0 \delta_0[a] = -\delta_0 d_0[a]$ means:

- if $[a]$ belongs to $\text{im } d_0$, $[a] = d_0[b]$, then $\delta_0[a] = -d_0 \delta_0[b]$, i.e., if $[a]$ represents an obstruction to the lift of an element $[b]$, its anomaly represents minus the obstruction to the lift of the anomaly of $[b]$,
- if $d_0[a] = 0$, then $d_0 \delta_0[a] = 0$, i.e., if $[a]$ can be lifted, then so does its anomaly $\delta_0[a]$,
- the anomaly in a bottom $[a]$ of K_0 that cannot be lifted is minus the anomaly of the corresponding obstruction, up to elements that can be lifted.

If we organize the space $E_0 \simeq K_0 = H(\bar{s})$ as $E_0 = F_0 \oplus E_1 \oplus d_0 F_0$, with $E_1 \simeq K_1$, we have shown that 1-loop contribution to the anomaly of an element in one of these subspaces belongs to the same subspace or to a subspace that stands to the right. Together with the last point of the previous list, this sums up the results for the elements of $H(\bar{s})$, i.e., for the obstructions and the bottoms contained in (9.19), (9.22) and (9.23) restricted to $r = 0$.

In the same way, these results for all r and s follow from the fact that δ_0 induces a well-defined differential (also called δ_0 in the following) in the spaces K_r , anticommuting with d_r , $\delta_0 : K_r \longrightarrow K_r$ with $\{\delta_0, d_r\} = 0$.

Indeed, suppose this result to be true for $K_0, \dots, K_{r-1}, d_0, \dots, d_{r-1}$. An element $[a]_r \in K_r$ satisfies $\bar{s}a = 0, da + \bar{s}c_1 = 0, \dots, dc_{r-1} + \bar{s}c_r = 0$. This implies $s^q a \circ \Gamma^\infty = \hbar b \circ \Gamma^\infty, da \circ \Gamma^\infty + s^q c_1 \circ \Gamma^\infty = \hbar f_1 \circ \Gamma^\infty, \dots, dc_{r-1} \circ \Gamma^\infty + s^q c_r \circ \Gamma^\infty = \hbar f_r \circ \Gamma^\infty$. Applying s^q gives to lowest order $\bar{s}b = 0, db + \bar{s}f_1 = 0, \dots, df_{r-1} + \bar{s}f_r = 0$, so that $[b]_r$ is well defined. Suppose now that $[a]_r = d_0[g_0]_0 + \dots + d_{r-1}[g_{r-1}]_{r-1}$. Anticommutativity of δ_0 with d_0, \dots, d_{r-1} then implies that $\delta_0[a]_r = 0$. Hence, δ_0 does not depend on the representative and is well defined in K_r . Finally, $d_r \delta_0[a]_r = d_r[b]_r = [df_r]_r$, while $\delta_0 d_r[a]_r = \delta_0[dc_r]_r$, and $s^q(dc_r) \circ \Gamma^\infty = s^q d(c_r \circ \Gamma^\infty + \hbar c' \circ \Gamma^\infty) = -d(s^q c_r \circ \Gamma^\infty + \hbar s^q c' \circ \Gamma^\infty)$, so that $\delta_0[dc_r]_r = -[d(f_r + \bar{s}c')]_r = -d_r \delta_0[a]_r$.

The results (9.19), (9.22) and (9.23) can then be summarized by the statement that an anomaly in one of the subspaces of the decomposition (9.26) must belong to the same subspace or to one that stands to the right; furthermore, the part of the anomaly of an element of F_i in F_i is minus the part of the anomaly of the corresponding element of $d_i F_i$ in $d_i F_i$.

In order to recover the results for elements of $H(\bar{s}|d)$, we define Δ_0 to be the equivalent of δ_0 for modulo d BRST cohomological classes, $\Delta_0[a] = [b]$, for $[a], [b] \in H(\bar{s}|d)$, where $\bar{s}a + dm = 0, \bar{s}m + du = 0, s^q a \circ \Gamma^\infty + d(m \circ \Gamma^\infty) = \hbar b \circ \Gamma^\infty, s^q m \circ \Gamma^\infty + d(u \circ \Gamma^\infty) = \hbar n \circ \Gamma^\infty$. Indeed, the map is well defined because the consistency condition implies to lowest order $\bar{s}b + dn = 0$, while if $a = \bar{s}c + dg$, we have $m = \bar{s}g + du$, so that $s^q a \circ \Gamma^\infty + dm \circ \Gamma^\infty = \hbar b \circ \Gamma^\infty$ gives $s^q(s^q c \circ \Gamma^\infty + dg \circ \Gamma^\infty + \hbar f \circ \Gamma^\infty) + d(s^q g \circ \Gamma^\infty + du \circ \Gamma^\infty + \hbar v \circ \Gamma^\infty) = \hbar b \circ \Gamma^\infty$, which implies $b = \bar{s}f + dv$ as it should.

The following properties are straightforward to check: $[\Delta_0, \mathcal{D}] = 0, l\Delta_0 = -\delta_0 l, i_0 \delta_0 = \Delta_0 i_0$. One says (see e.g. [116], Chapter VIII.9) that (Δ_0, δ_0) is a mapping of the exact couple $(H(\bar{s}|d), H(\bar{s}))$.

The previous result, that δ_0 induces well defined maps in the spaces of the spectral sequence anticommuting with the differentials d_r , follows directly from the way the spectral sequence is associated to an exact couple. The relation between (9.11), (9.16) at $l = 0$ and (9.22), (9.23) is summarized by $i_0 \delta_0 = \Delta_0 i_0$; the relations between (9.11), (9.16) at different values of l are summarized by $[\Delta_0, \mathcal{D}] = 0$; finally, the relation between (9.16) at $l = r$ and (9.19) is summarized by $l\Delta_0 = -\delta_0 l$.

Note that in this case, we have furthermore the property that Δ_0 is a differential, $\Delta_0^2 = 0$.

Remark: It follows from the above analysis that the relevant property of the differentials d_r is $\{\delta_0, d_r\} = 0$. This means that analogous results that constrain the anomalies to belong to particular subspaces of $H(\bar{s})$ or $H(\bar{s}|d)$ can be derived if one can find maps $\lambda_0 : H(\bar{s}) \rightarrow H(\bar{s})$, respectively $\Lambda_0 : H(\bar{s}|d) \rightarrow H(\bar{s}|d)$ such that $[\delta_0, \lambda_0] = 0$, respectively $[\Delta_0, \Lambda_0] = 0$.

The discussion in section on the length of descents and lifts of BRST cohomological classes and their anomalies in this and the previous subsection does not rely on the use of the extended antifield formalism. It can be done along the same lines in the standard set-up as long as one assumes the quantum theory to be anomaly free and stable, so that the standard Zinn-Justin equation $\frac{1}{2}(\Gamma, \Gamma) = 0$ holds.

10 Application 2: non renormalization of the Y-M gauge anomaly

We now show that the anomalous master equation for Yang-Mills theories discussed in [55, 59, 65] can be viewed as a particular case of the anomalous master equation of the extended antifield formalism. Then, we discuss how the Adler-Bardeen theorem for the non abelian gauge anomaly [117, 55, 118, 119, 120] can be understood as a direct consequence of the fact that the length of the descent of the gauge anomaly is 4, while the length of the descent of all the other cohomological classes coupled to the action is 0. These considerations are purely cohomological, so that they do not depend on the way the gauge is fixed or on power counting restrictions. Furthermore, this approach to the Adler-Bardeen theorem does not require the use of the Callan-Symanzik equation or assumptions on the beta functions of the theory.

In the case of standard Yang-Mills theory, it is sufficient for our purpose to couple the local BRST cohomology classes in ghost number 0 and ghost number 1, because this will be enough, under some assumptions stated explicitly below, to guarantee stability and to control the anomalies. The starting point action contains from the beginning a coupling to the Adler-Bardeen anomaly as in [55, 59, 65], with additional couplings to possibly higher dimensional gauge invariant operators, if one does not want to restrict oneself to the power counting renormalizable case [42]. More precisely, the starting point is the action

$$\begin{aligned} S_\rho = & \int d^4x \left[-\frac{1}{4g^2} F_I^{\mu\nu} F_{\mu\nu}^I + L_{\text{matter}}^{\text{kin}}(y^i, D_\mu y^i) \right] \\ & + \int d^4x \left[-D_\mu C^I A_I^{*\mu} + C^I T_{Ij}^i y^j y_i^* - \frac{1}{2} C^I C^J f_{JI}^K C_K^* \right] \\ & + g^i \int d^4x \mathcal{O}_i + \mathcal{A}\rho, \end{aligned} \quad (10.1)$$

satisfying the master equation

$$\frac{1}{2}(S_\rho, S_\rho) = 0. \quad (10.2)$$

The Lagrangian $L_{\text{matter}}^{\text{kin}}(y^i, D_\mu y^i)$ is the gauge invariant extension of the kinetic terms for the matter fields y^i . For simplicity, we assume the gauge group to be $SU(3)$. The \mathcal{O}_i are gauge invariant local functions built out of the field strengths $F_{\mu\nu}^I$, the matter fields y^i and their covariant derivatives such that the $\int d^4x \mathcal{O}_i$ (which can, but need not, be assumed to be power counting renormalizable) and $\int d^4x -\frac{1}{4g^2} F_I^{\mu\nu} F_{\mu\nu}^I$ are linearly independent even when the gauge covariant equations of motions hold. Finally, $\mathcal{A} = \int \text{Tr} [Cd(AdA + \frac{1}{2}A^3)]$ is the Adler-Bardeen gauge anomaly, g is the gauge coupling constant, g^i are coupling constants for the other gauge invariant operators, while ρ is a Grassmann odd coupling constant with ghost number -1 for the Adler-Bardeen anomaly. In this particular case, $\Delta_c = 0$. This can be traced back to the fact that all representatives of the local BRST cohomological classes in ghost number 0 and 1 can be chosen to be independent of the antifields.

The gauge is fixed by introducing the cohomologically trivial non minimal sector consisting of the antighost \bar{C}^I and the Lagrange multiplier B^I and their antifields. One adds to the action (10.1) the term $\int d^4x \bar{C}_I^* B^I$ and chooses an appropriate gauge fixing fermion Ψ , which is used to generate an anticanonical transformation in the fields and antifields such that the propagators of the theory are well defined. The gauge fixing is irrelevant for the cohomological considerations below.

For the question of stability and anomalies, we have to analyze the cohomology $H^{0,4}(s_\rho|d)$ and $H^{1,4}(s_\rho|d)$ in the space of functions in the couplings g, g^i, ρ with coefficients that are Lorentz invariant polynomials in the dx^μ , the fields, the antifields and their derivatives.

In order to compute this cohomology, we decompose, as in [55], both the BRST differential s_ρ and the local forms into parts independent of ρ and parts linear in ρ . Explicitly, $s_\rho = s_0 + s_1$, where s is the standard BRST differential associated to the solution $S_{\rho=0}$ of the master equation, while $s_1 = (\mathcal{A}\rho, \cdot)$. The ρ independent part of the cocycle condition $s_\rho \omega(\rho) + d\eta(\rho) = 0$ in form degree 4 gives (see e.g. [33] for a review) $\omega(0) = \alpha(g, g^i) d^4x F_I^{\mu\nu} F_{\mu\nu}^I + \alpha^j(g, g^i) d^4x \mathcal{O}_i + s(\cdot) + d(\cdot)$. This implies $\omega(\rho) = \alpha(g, g^i) d^4x F_I^{\mu\nu} F_{\mu\nu}^I + \alpha^j(g, g^i) d^4x \mathcal{O}_i + \omega'_1 \rho + s_\rho(\cdot) + d(\cdot)$. Because $d^4x F_I^{\mu\nu} F_{\mu\nu}^I$ and $d^4x \mathcal{O}_i$ are also s_1 closed and $\rho^2 = 0$, the cocycle condition reduces to $s\omega'_1 + d(\cdot) = 0$, where the ghost number of ω'_1 is 1. It follows that $\omega'_1 = \lambda(g, g^i) \text{Tr} [Cd(AdA + \frac{1}{2}A^3)] + s(\cdot) + d(\cdot)$. Hence, the general solution of the consistency condition in the space of local functionals in ghost number 0 is given by

$$\alpha(g, g^i) \frac{\partial S_\rho}{\partial g} + \alpha^j(g, g^i) \frac{\partial S_\rho}{\partial g^i} + \frac{\partial S_\rho}{\partial \rho} \rho \lambda(g, g^i) + (S_\rho, \Xi_\rho) \quad (10.3)$$

for some local functional Ξ_ρ in ghost number -1 . This implies that the theory is stable.

Similarly, in ghost number 1, the ρ independent part of the cohomology gives as only anomaly candidate the Adler-Bardeen anomaly. There could however be a ρ linear non trivial contribution to the anomaly, because the cohomology of s in form degree 4 and ghost number 2 is not necessarily empty (see [33], section 12.4), contrary to the claim in [55]. More precisely, to each x^μ -independent, gauge and Lorentz invariant non trivial conserved current $j_\Delta = j_\Delta^\mu \epsilon_{\mu\nu_1\nu_2\nu_3} dx^{\nu_1} dx^{\nu_2} dx^{\nu_3}$, there corresponds the cohomological class $V_\Delta^{2,4} = j_\Delta [\text{Tr} C^3]^1 + \text{antifield dependent terms}$ in $H^{2,n}(s|d)$, with $s[\text{Tr} C^3]^1 + \text{Tr} C^3 = 0$. (There could in principle be another type of antifield dependent cohomology classes in exceptional situations [33], which we exclude from the present considerations).

If there are such non trivial currents j_Δ , we have to change our starting point and also couple the “anomaly for anomaly candidates” $V^{2,4}$ from the beginning with couplings in ghost number -2 . But then the cohomology of s in ghost number 3 becomes relevant. There are plenty of such classes, for instance classes of the form $d^4x I \text{Tr} C^3$, where I are invariant functions built out of the field strengths, the matter fields and their covariant derivatives. In this way, one is led to use the full extended antifield formalism as described in section 5 with all BRST cohomology classes in positive ghost number coupled from the beginning. In the case where the algebra of

the non trivial symmetries associated to the currents j_Δ is non abelian, the operator Δ_c will be non vanishing at the classical level and involve in particular the structure constants of this algebra.

Another possibility is to try to show that the anomaly candidates $V_\Delta^{2,4}$ do not effectively arise in the theory, by using higher order cohomological restrictions: as in the proof of the absence of similar instabilities in the presence of abelian factors for standard Yang-Mills theories in section 3 one couples with external fields gauge invariant functions that break the symmetries associated to the currents j_Δ . In this way, one eliminates the currents j_Δ and the associated anomaly for anomaly candidates $V_\Delta^{2,4}$ from the extended theory. At the end of the computations, the external fields can be put to zero.

Because this discussion is not central to the argument below, we will simply assume here that the observables \mathcal{O}_i are such that there are no non trivial currents j_Δ and thus no anomaly for anomaly candidates $V_\Delta^{2,4}$ in the theory. The general solution of the consistency condition in ghost number 1 is then given by

$$\frac{\partial S_\rho}{\partial \rho} \sigma(g, g^i) + (S_\rho, \Sigma_\rho), \quad (10.4)$$

for some local functional Σ_ρ in ghost number 0.

By standard arguments, using in addition the same reasoning as in section 6.2.2, it follows from (10.3) and (10.4) that the model is renormalizable and that the renormalized generating functional for 1 particle irreducible vertex functions Γ_ρ satisfies

$$\frac{1}{2}(\Gamma_\rho, \Gamma_\rho) + \frac{\partial^R \Gamma_\rho}{\partial \rho} \hbar \sigma(g, g^i) = 0, \quad (10.5)$$

where $\sigma(g, g^i)$ is a formal power series in \hbar . Hence, in this case $\Delta^\infty = \frac{\partial^R}{\partial \rho} \hbar \sigma(g, g^i)$ and the quantum BRST differential is $s^q = (\Gamma_\rho, \cdot) - \hbar \sigma(g, g^i) \partial^L \partial \rho$.

Let us now investigate the renormalization of the operators $d^4x F_I^{\mu\nu} F_{\mu\nu}^I$ and $d^4x \mathcal{O}_i$. According to the classification in section 9.1, they are both of the type $e_{\alpha_0}^0$, because they are non trivial bottoms in maximal form degree 4, while the Adler-Bardeen anomaly $\text{Tr} [Cd(AdA + \frac{1}{2}A^3)]$ is of the type $e_{\alpha_4}^4$, as it descends to the non trivial bottom $\text{Tr} C^5$. Because there is no $e_{\alpha_0}^0$ in ghost number 1 (and form degree 4) and no $h_{i_r}^r$ in form degree 3 and ghost number 1, equation (9.11) implies that the lowest order contribution to the anomaly in the renormalization of $d^4x F_I^{\mu\nu} F_{\mu\nu}^I$ and of $d^4x \mathcal{O}_i$ is s_ρ exact and can thus be absorbed through a BRST breaking counterterm added to these operators. This reasoning can be pushed to all orders, with the result that one can achieve, through the addition of suitable counterterms,

$$s^q[d^4x F_I^{\mu\nu} F_{\mu\nu}^I \circ \Gamma_\rho] = 0, \quad (10.6)$$

$$s^q[d^4x \mathcal{O}_i \circ \Gamma_\rho] = 0. \quad (10.7)$$

If we now apply $-\frac{g}{2} \frac{\partial}{\partial g}$, respectively $\frac{\partial}{\partial g^j}$ to (10.5), we get on the one hand,

$$s^q[\int d^4x -\frac{1}{4g^2} F_I^{\mu\nu} F_{\mu\nu}^I \circ \Gamma_\rho] + \frac{\partial^R \Gamma_\rho}{\partial \rho} \hbar [-\frac{g}{2} \frac{\partial \sigma(g, g^i)}{\partial g}] = 0, \quad (10.8)$$

$$s^q[\int d^4x \mathcal{O}_i \circ \Gamma_\rho] + \frac{\partial^R \Gamma_\rho}{\partial \rho} \hbar[\frac{\partial \sigma(g, g^i)}{\partial g^j}] = 0. \quad (10.9)$$

Comparing on the other hand with the integrated versions of (10.6) and (10.7), we deduce that

$$\frac{\partial \sigma(g, g^i)}{\partial g} = 0, \quad \frac{\partial \sigma(g, g^i)}{\partial g^j} = 0, \quad (10.10)$$

which is crucial in the proof of the Adler-Bardeen theorem (see e.g. section 6.3.2 of [47]).

11 Conclusion

Starting from the standard antifield formalism, one first computes a basis for the representatives of the local BRST cohomology of the theory. The representatives of those classes that are not already contained in the standard solution of the master equation are coupled with new coupling constants. The extended formalism can then be constructed perturbatively and involves the antibracket algebra of these representatives. The extended master equation $\frac{1}{2}(S, S) + \Delta_c S = 0$ is stable by construction, because the associated differential $\bar{s} = (S, \cdot) + \Delta_c^L$ takes into account higher order cohomological restrictions encoded in Δ_c^L .

The extended master equation can be constructed consistently in two particular cases. The first case is the extended antifield formalism of [31], where only the local BRST cohomological classes in strictly negative ghost numbers have been coupled. This is the relevant formalism to control the algebra of the non trivial generalized global symmetries of the theory. In the second case, only the classes in zero and positive ghost numbers are coupled. The formalism then allows to control the renormalization, where only the gauge symmetries and no global symmetries are taken into account.

The stability of the formalism guarantees renormalizability in the modern sense for generic gauge theories. The renormalized effective action Γ^∞ satisfies the extended master equation $\frac{1}{2}(\Gamma^\infty, \Gamma^\infty) + \Delta^\infty \Gamma^\infty = 0$, where the differential Δ^∞ is a deformation of the differential Δ_c and encodes all the information about the anomalies. The associated quantum BRST differential $s^q = (\Gamma^\infty, \cdot) + (\Delta^\infty)^L$ is well defined and reduces to \bar{s} in the classical limit.

The standard master equation, both on the classical and the quantum level, expresses gauge invariance of the original theory. The extended master equation allows for a breaking of gauge invariance, even on the classical level. This happens if Δ_c does not vanish. The important question of gauge invariance on the quantum level for the original theory can be answered by putting to zero the additional couplings after renormalization. If this can be done and Δ^∞ then vanishes, the quantum theory is gauge invariant in the sense that the standard Zinn-Justin equation for the effective action holds.

Controlling the symmetry and anomaly structure of the theory, by first computing the complete classical cohomology, then constructing a stable formalism and finally quantizing, i.e., computing Γ^∞ and Δ^∞ , without any assumptions about anomalies needed, could be called “cohomological renormalization”, in contrast to “algebraic renormalization”, where for a given theory, stability has to be checked and absence of anomalies is required.

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